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The Expected Length of a Biased Random Walk

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ABSTRACT

In this note, we find the expected length of a biased random walk on a linear array of points and connecting segments. The endpoints serve as traps or absorbing boundaries for the walk. The probabilities of moving from any interior point to its two nearest neighbors are p and $1 - p$, $0 < p < 1$. When $p \neq \frac{1}{2}$ the walk is said to be biased. The work proceeds from a specific case to the general result suggested by the initial computations.

We show that the expected length of a walk beginning at the *i-th* point in an array of n points is given by

$$E(n, i) = A + Bi + C\left(\frac{p}{1-p}\right)^{n-1}$$

where the values of the constants A,B and C are found in terms of $p \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1)$.

INTRODUCTION

It is so obvious that it becomes a cliché to say that old mathematics finds new uses. Nevertheless, the idea deserves to be noted. Arising in classical probability theory, the problem of a random walk from location to location in physical space or from state to state in some abstract space finds extensive use in contemporary applied mathematics.

Random walks model diffusion processes of interest to physicists (Feynman, 1963) and biologists (Murray, 1993). Their relevance to electrical circuits is well known (Doyle and Snell, 1984), and they find application in computer science (Kruse, Leung, and Tondo, 1991). The random walk which we shall present is one dimensional.

Let us consider the points $x = 0, 1, 2, \ldots, n$ on a coordinate axis. The endpoints $x = 0$, $x = n$ serve as traps for a random walk on the array in which each step has unit length (Figure 1). The probability of moving from x to $x + 1$ for $x = 1,2,3,\ldots, n-1$ is p where $p \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1)$. The probability of each step toward $x = 0$ (from x to $x - 1$) is $1 - p$. The walk is said to be biased in the direction determined by the greater of p and $1 - p$.

Although the random walk is easy to describe, it is also quite easy to ask difficult questions about such a stochastic process. There was one particular question which we thought that we should be able to answer with methods and ideas no more sophisticated than those which are developed in an undergraduate, introductory probability and statistics course. That question was "What is the expectation value for the length of the random walk which we have described?"

We have done our mathematics with an "economy of means. " Only after obtaining our results did we learn that the problem is a variant of "The Gambler's Ruin." (Feller, 1968). Since we are unable to claim anything in the way of an original result, we take comfort in citing a remark made by the distinguished mathematician R. Hamming:

FIGURE 1. The array for the random walk.

"The best thing that ever happened in the world is burning the library in Alexandria because it removed a millstone from around people's necks." Our reading of Hamming's statement is that originality is not everything in mathematics and too strict an insistence upon it stifles creativity (Albers and Alexanderson, 1985). We are of the opinion that our work remains interesting as a straightforward, intuitive attack upon a nontrivial problem most appropriate to undergraduate probability and statistics.

In our note, we compute the expected length of the random walk as a function of n (where there are $n + 1$ points) and of i $\in \{0, 1, 2, ..., n\}$, the coordinate of the starting point of the walk. We denote the expected value by $E(n, i)$. We provide a path of discovery leading from a specific example to a general result. The computations involve the method of successive differences, a technique which is somewhat neglected in undergraduate mathematics these days.

A SPECIAL CASE

First, we consider a special case in hopes that its solution will direct us toward the general result. We take that to be the case for $p = \frac{2}{3}$

From the interior point for which $x = i$, the probability of a step to the left (to i - 1) is $\frac{1}{3}$ and the probability of a step to the right (to $i + 1$) is $\frac{2}{3}$. Since one step takes the walk to a new "starting point", we can write a recursion relation for the expected length

$$\begin{aligned} E(n, i) &= \frac{1}{3}(1 + E(n, i-1)) + \frac{2}{3}(1 + E(n, i+1)) \\ &= \frac{1}{3} E(n, I-1) + \frac{2}{3} E(n, i+1) + 1 \ . \end{aligned} \tag{1}$$

Then a bit of algebraic manipulation yields

$$3\, E(n, i) = E(n, i-1) + 2\, E(n, i+1) + 3 \ . \tag{2}$$

At the endpoints $i = 0$ and n, we have $E(n, 0) = E(n, n) = 0$. The endpoint values are our boundary conditions.

Solving the system of linear equations defined by Equation 2 with $i = 1,2,3,..., n$ -1 and the boundary conditions for $n = 3, 4, 5$ yields

$E(3,0) = 0$, $E(3,1) = \frac{15}{7}$, $E(3,2) = \frac{12}{7}$, $E(3,3) = 0$;

$E(4,0) = 0$, $E(4,1) = \frac{17}{5}$, $E(4,2) = \frac{18}{5}$, $E(4,3) = \frac{11}{5}$, $E(4,4) = 0$; and

$E(5,0) = 0$, $E(5,1) = \frac{147}{31}$, $E(5,2) = \frac{174}{31}$, $E(5,3) = \frac{141}{31}$, $E(5,4) = \frac{78}{31}$, $E(5,5) = 0$.

$$
\begin{array}{ccccccccccc}
0 & & \frac{147}{31} & & \frac{174}{31} & & \frac{141}{31} & & \frac{78}{31} & & 0 \\
& \vee & & \vee & & \vee & & \vee & & \vee & \\
& \frac{147}{31} & & \frac{27}{31} & & \frac{-33}{31} & & \frac{-63}{31} & & \frac{-78}{31} & \\
& & \vee & & \vee & & \vee & & \vee & & \\
& & \frac{-120}{31} & & \frac{-60}{31} & & \frac{-30}{31} & & \frac{-15}{31} & & \\
& & & \vee & & \vee & & \vee & & & \\
& & & \frac{-60}{31} & & \frac{30}{31} & & \frac{15}{31} & & & \\
& & & & \vee & & \vee & & & & \\
& & & & \frac{-30}{31} & & \frac{-15}{31} & & & & \\
& & & & & \vee & & & & & \\
& & & & & \frac{15}{31} & & & & &
\end{array}
$$

FIGURE 2. The Successive Differences.

Seeking a pattern for $E(n, i)$ as n varies, we next examine the successive differences $E(n, i) - E(n, i-1)$ for $n = 3,4$, and 5. The tableau of differences from $n = 5$ is given in Figure 2.

These computations suggest that

$$E(n, i) = A + Bi + C2^{n-i}$$

with the constants to be determined.

Equation 2 implies that $B = -3$ while $E(n, 0) = E(n, n) = 0$ implies that $A = \dfrac{3n2^n}{2^n - 1}$ and $C = \dfrac{-3n}{2^n - 1}$.

Thus

$$E(n, i) = \frac{3n2^n}{2^n - 1} - 3i - \frac{3n2^{n-i}}{2^n - 1} . \qquad (3)$$

A bit of algebraic manipulation shows that Equation 3 satisfies Equation 2 and the boundary conditions. If a second function $F(n, i)$ also satisfies Equation 2 and the boundary conditions, then $G(n, i) = E(n, i) - F(n, i)$ satisfies

$$G(n, i) = \frac{1}{3}G(n, i-1) + \frac{2}{3}G(n, i+1)$$

for $i \in \{1, 2, 3, ..., n - 1\}$ and $G(n, 0) = G(n, n) = 0$. It follows by reasoning analogous to that for discrete harmonic functions agreeing at all boundary points that $G(n, i) \equiv 0$ and that $E(n, i) \equiv F(n, i)$ (Boyd and Raychowdhury, 1989; 1992). Therefore, $E(n, i)$ in Equation 3 is the unique solution for $p = \frac{2}{3}$

THE GENERAL CASE

Let $1 - p$ and p replace $\frac{1}{3}$ and $\frac{2}{3}$ respectively, in Equation 1. Then we are led to consider

$$E(n, i) = (1 - p)E(n, i - 1) + pE(n, i + 1) + 1. \tag{4}$$

Let us assume that $E(n, i) = A + Bi + C\left(\frac{p}{1-p}\right)^{n-1}$ and pursue the consequences of that assumption.

Letting $E(n, i)$ take its assumed form in Equation 4 implies that $B = \frac{1}{1-2p}$. Letting $i = 0$ and $i = n$ in the assumed form yields

$$A + C\left(\frac{p}{1-p}\right)^{n} = 0 \text{ and}$$

$$A + C = \frac{-n}{1-2p} .$$

Solving these equations , we find that

$$C = \left(\frac{n}{1-2p}\right) \Big/ \left[\left(\frac{p}{1-p}\right)^{n}\right] - 1 \text{ and}$$

$$A = -\left(\frac{p}{1-2p}\right)^{n}\left(\frac{n}{1-2p}\right) \Big/ \left[\left(\frac{p}{1-p}\right)^{n} - 1\right] .$$

We observe that A, B, and C reduce to the values in the special case when $p = \frac{2}{3}$.

Algebraic manipulation shows that the assumed solution does indeed satisfy Equation 4 and the boundary conditions. A generalization of the uniqueness argument in the special case holds as well.

Thus

$$E_{(n,\,i)} = A + Bi + C\left(\frac{p}{1-p}\right)^{n-i}$$

is the general solution for $p \in \left(0, \frac{1}{2}\right) \cup \left(\frac{1}{2}, 1\right)$ with A, B, C having the values given above.

DISCUSSION AND EXTENSIONS

The reader may verify that $\lim_{p \to o^+} E(n, i) = i$ and $\lim_{p \to 1^-} E(n, i) = n - i$ for $i \in \{1, 2, 3, ..., n-1\}$ thus justifying that $E(n, i) = i$ and $E(n, i) = n - i$ for $p = 0$ and $p = 1$, respectively. These results on the interior of the array are as expected.

When $p = \frac{1}{2}$, Equation 4 becomes

$$E(n, i) = \frac{1}{2}(E(n, i-1) + E(n, i+1)) + 1 \ . \tag{5}$$

As we have previously shown (Boyd and Raychowdhury, 1991),

$$E(n, i) = n i - i^2$$

uniquely satisfies Equation 5 for $E(n, 0) = E(n, n) = 0$.

LITERATURE CITED

Albers, D.J. and G.L. Alexanderson,Editors. 1985. Mathematical People. Birkhauser, Boston. p244.

Boyd, J.N. and P.N. Raychowdhury. 1989.Discrete Dirichlet Problems, Convex Coordinates, and a Random Walk on a Triangle. The College Mathematics Journal 20:385-392.

Boyd, J.N. and P.N. Raychowdhury. 1991. Fair Fare Functions. Pi Mu Epsilon Journal 9:242-246.

Boyd, J.N. and P.N. Raychowdhury. 1992. Complete Fare Functions. Mathematics and Computer Education 26:230-238.

Doyle, P.G. and J.L. Snell. 1984. Random Walks and Electrical networks. Mathematical Association of America, Washington, D.C.

Feller, W. 1968. An Introduction to Probability Theory and Its Applications. Vol. I (Third Edition), John Wiley, New York. pp 342-349.

Feynman, R.P., R.B. Leighton, and M. Sands. 1963. Lectures on Physics, Vol. I, Addison-Wesley, Reading, MA. pp 65-68.

Kruse, R.L., B.P. Leung, and C.L. Tondo. 1991. Data Structures and Program Design, Prentice Hall, Englewood Cliffs, NJ. pp 88-89.

Murray, J.D. 1993. Mathematical Biology (second edition), Springer Verlag, New York.pp 232-236.