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Network Contraction in R^{n+1}

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ABSTRACT

This paper presents the principles and essential elements of DINER^{n+1}, a simple scheme for minimizing the storage requirements of a network of observed data in R^{n+1}, without significant loss of the informational value of the data set. DINER^{n+1} partitions a subject network into a set of contiguous atomic sub-networks, all of which share a common interpolation functional form. Generalizability and extendability of such a network contraction scheme to a wide variety of contexts is of considerable interest in most scientific and technological applications.

INTRODUCTION

Let $D_N{}^{n+1} = \{(x_i, d_i) R^{n+1} \mid x_i \varepsilon R^n, d_i \varepsilon R; i = 1,2,...,N\}$ denote a set of N discrete points in R^{n+1}, which represents an observation of some system. The real numbers, d_i, represent some information about the system at space-time locations $x_i \varepsilon R^n$. Each point-value pair: (x_i, d_i), may then be called a digital information node; in which case $D_N{}^{n+1}$ is an N-node digital information network in R^{n+1}.

Computational physics, which deals largely with simulation modeling of physical systems, employs such digital information networks to model physical systems. The number N, of nodes in such network models required to realistically capture the essence of a practical physical system is usually very large; strategies must be devised to significantly minimize N, without losing the essential characteristics of the physical system being modeled.

The information super-highway bandwagon, which is currently parading everywhere , promising instant information on any and everything, from any and everywhere, to any and everyone, must be ready to store and fetch very large amounts of information across a network of information consumers/producers. Its practicality demands, among other things, efficient schemes for minimalist information representation.

This paper proposes and demonstrates a simple, reasonably accurate, fast and efficient scheme, called DINER^{n+1}, for minimizing digital information networks in R^{n+1}.

NETWORK BASICS

A network is generally defined as a domain of elements (called nodes) that are connected by links (called branches or arcs). A network node may be a source, sink or transhipment point for the network's information stream. A branch, on the other hand, is like a conduit for information flow, but one which may also process (that is, manipulate) the information as it flows along the branch. The nature of a network's information stream depends on the system being modeled by the net-

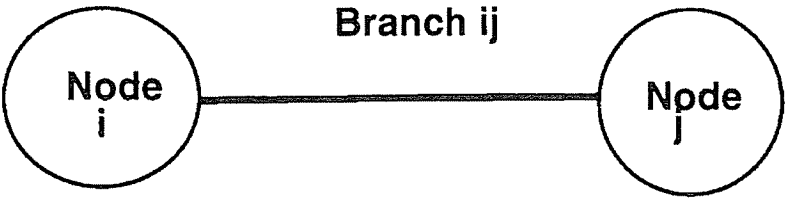

FIGURE 1. Schema of a Network Branch.

work, and the number and distribution of the nodes and branches in a network depend on the complexity of the subject system.

Regardless of the space-time dimensionality of a network, each network branch is always characterized by a pair of nodes, one of which may be called the head- and the other the tail-node (see Figure 1). The branch itself can be modeled as a transformer whose input is the head-node information and whose output is the tail-node information. A collection of connected branches linking two nodes that are not next neighbors is called a path (or a tree); it may be considered to be a macro-branch, and, therefore, a complex (usually non-linear) transformer of the information at its end nodes. In this paper the word branch refers to either a simple or a macro branch. Further, a branch interpolation function is defined as the functional form of a branch's transformer. Given the head and/or tail node, a branch interpolation function should be able to present, on demand, any intermediate nodes on the branch, and, therefore, all the information of the sub-network represented by the branch.

A network may be partitioned into contiguous sub-networks. This is usually done by selecting some subset of the network's nodes as anchor nodes, and creating a new network of such anchor nodes. Each sub-domain of the original network, bounded by anchor nodes and anchor-node-branches, is then called a partition of the original network. Let:

$$D_N^{n+1} = \sum_{j=1}^{L} \left(S_M^{n+1} \right)_j \qquad (1)$$

where: $L < N$; $(S_M^{n+1})_j = \{ (x_k, d_k) \; R^{n+1} \mid x_k \in R^n, d_k \in R; \; k = 1,2,....,M \}$; and, desirably, $M << N$. $(S_M^{n+1})_j$ is called a partition or sub-network of D_N^{n+1}. In the same way that a branch interpolation function was defined, a partition interpolation function may be defined, using the bounding anchor nodes, such that the function interpolates all nodes of the original network that are resident in the partition.

THE NETWORK CONTRACTION PROBLEM

The digital information network contraction problem may be stated simply as follows: Given D_N^{n+1}, find the smallest network:
$C_M^{n+1} = \{(x_k, d_k) \; R^{n+1} \mid x_k \in R^n, d_k \in R; k = 1,2,...,M\}$, such that: $M < N$,

and C_M^{n+1} ($\subseteq D_N^{n+1}$) captures all the informational value of D_N^{n+1}. This is a fundamental problem in computational physics, image coding/decoding, digital data compression, and similar emergent signal processing technologies of modern information science. Although related, the above problem is quite different from the basic problem of "Minimal Networks Theory" (Ivanov and Tuzhilin 1994) encountered in minimal spanning tree and shortest route analyses.

Several significant but independent strides, (Schumaker and Webb, 1994), (Meyer, 1993), (Barnesly, 1993) and (Thompson and Weatherill, 1993), have already been made towards solving this network contraction problem in several application contexts, but all such efforts are for the most part disjointed and difficult to extend across application areas. For example, a digital data compression scheme seems to bear no relationship whatsoever to a computational physics grid technology for, say, fluid flow simulation, and vice-versa. Yet, each of those two applications is precisely an exercise in network contraction. A unified solution which is extendable within and across application contexts is clearly needed.

NETWORK CONTRACTION STRATEGIES

If D_N^{n+1} contained several spatial and/or temporal redundancies that are not necessary for specifying the network's informational value, then removal of all such redundancies should minimize D_N^{n+1}, without loss of its informational value. Alternatively, and perhaps even concurrently with redundancy removal, it may be possible to construct a new representation format for the informational value of D_N^{n+1}, which makes significantly less demand on storage resources. For instance, instead of the N nodes of D_N^{n+1} one could seek to represent the network's information with just a subset of the network's boundary nodes plus an interpolation functional form which can compute any of the other nodes of D_N^{n+1}, on demand. The storage required for such boundary nodes and for the interpolation functional form should be significantly less than was needed for the original D_N^{n+1} network. This latter representation, when realizable, is the ideal (or optimal) contraction of D_N^{n+1}. But it is an ideal that is often unrealizable for most systems of scientific and technological interest.

A key reason for the unrealizability of the optimal (that is, single functional) representation of system information networks lies in the fact that, except for the simplest and usually trivial systems, it is rather difficult to describe a system using a single functional form. Usually an infinite set (or a very large collection) of functional forms is needed to represent most systems of any practical scientific and technological interest. However, it may be possible, in the manner discussed earlier in this paper, to partition a D_N^{n+1} network into a set of sub-networks, each of which may be more amenable to optimal contraction. Among the sub-networks, $(S_M^{n+1})_j$, it may be further possible to identify some that are similar, that is, they have identical functional forms but differ in boundary node and/or function coefficient values. Such similar sub-networks are said to be redundant. Network redundancies can be removed by representing the redundant sub-networks as multiple instances of a single parametric sub-network. Partitioning, such as is described above, appears to be much more practicable than ideal contraction. It is the method of choice for most currently popular network contraction methods, such as wavelet schemes, (Meyer, 1993) and fractal compression methods, (Bar-

nesly, 1993), and (Jeff, 1994). And such partitioning can be accomplished with readily available schemes such as Delaunay triangulation.

Within the context of network contraction, as has been discussed above, a contraction ratio parameter Γ, may be defined as follows:

$$\Gamma = \text{(Original Network Storage)/(Contracted Network Storage)} \qquad (2)$$

The objective of a practical network contraction scheme is to maximize Γ; and a general strategy is to construct a feasible partitioning scheme to decompose a given network into a set of sub-networks, such that simple functional forms may be found to model each of the partitions. Successful application of these decomposition methods will depend on the partitioning strategy in use, as well as on the partition interpolation functional forms possible.

DINER^{n+1}

In light of the above discussions, a simple network contraction scheme will now be proposed. Called DINER^{n+1}: Digital Information Network Encoder in R^{n+1}, its objective is to partition a given D_N^{n+1} into a set of "atomic" sub-networks, each of which is then optimally represented with the same partition interpolation functional form. Such atomic partitions are extracted using the LBM algorithm (discussed below) to mark a subset of the nodes of D_N^{n+1} as anchor nodes. The network C_M^{n+1}, $(M < N)$, of boundary and interpolant anchor nodes thus obtained is a contraction of D_N^{n+1}; it is a network of anchored atomic partitions. The DINER^{n+1} process consists of the following steps:

(a) Distinguish the nodes of D_N^{n+1} into two major types: boundary (those which define the boundaries of the system being modeled) and interpolant (all other nodes of the system).

(b) Apply the LBM anchoring algorithm to identify and mark a subset of the boundary nodes as anchor nodes;

(c) Apply the LBM anchoring algorithm to identify and mark interpolant anchor nodes, if any; and

(d) Represent each atomic partition of D_N^{n+1} with the predictor-corrector partition interpolation functional form,

$$(L_j^{n+1} + E_j^{n+1}) \text{ such that:}$$

$$d_j^{n+1} = L_j^{n+1} + E_j^{n+1} \qquad (3)$$

where d_j^{n+1}, is the informational value of the jth node in the partition.

The predictor L_j^{n+1}, is a linear interpolation functional anchored on the partition's boundary nodes. In general, for any D_N^{n+1}, L_j^{n+1} is the functional of a uniform geometrical form bounded by (n+1) anchor nodes. For D_N^2, L_j^2 is a straight line joining the two anchor nodes; for D_N^3, L_j^3 is a planar surface connecting three anchor nodes; and for D_N^4, L_j^4 is a tetrahedral volume with four anchor nodes.

The corrector E_j^{n+1}, is a generally non-linear interpolation functional, which is used either to fit (in the case of a known d_j^{n+1} distribution), or to predict (in the

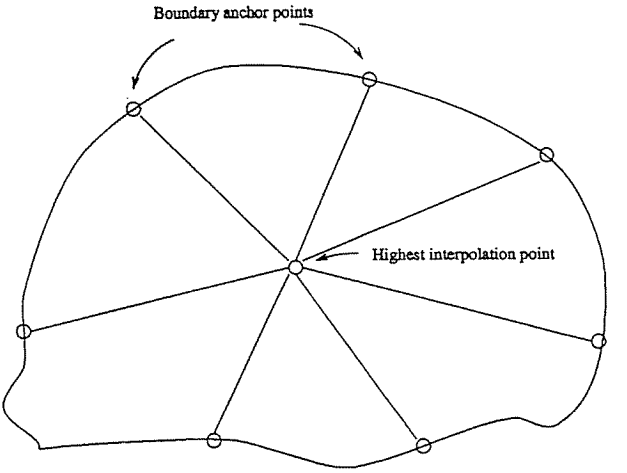

FIGURE 2. A typical LBM prtitioning of a D^3 network

case a simulation model), the error-distribution ($d_j^{n+1} - L_j^{n+1}$) . Because the boundary nodes of an atomic partition are always captured exactly, E_j^{n+1} is always zero-valued at an atomic partition boundary. Frank (1987) discusses several general fitting methods that could be used to define E_j^{n+1}.

L_j^{n+1} and E_j^{n+1} are each atomic partition invariant in R^{n+1}. That is, any atomic partition of D_N^{n+1} has exactly the same L_j^{n+1} and E_j^{n+1} functional forms. This fact is exploited in $DINER^{n+1}$, as a redundancy removal mechanism, to increase the network contraction ratio: Γ

LINEAR BASE MINMAX (LBM) ANCHORING ALGORITHM

Given a set of nodes, the LBM anchoring algorithm used to identify and mark the set's anchor nodes is implemented as follows:

First, identify one of the boundary nodes in the set as the source (or inflow) node, and another one (usually the one spatially and/or temporally farthest removed from the source node) as the sink (or outflow) node. The choice of source and sink nodes may be quite arbitrary, particularly if the subject set of nodes does not describe a flow system. The source and sink nodes are marked as the first set of anchor nodes. Then, using the current set of anchor nodes as the boundary nodes, and all the other nodes as interpolant nodes, locate the node in the set which has the maximum informational value, greater than the informational value of any of the boundary anchor nodes. If such a node exists, mark it as an anchor node and connect it with branches to each of the other anchor nodes, thereby partitioning the given set into sub-networks (see Figure 2). If such a maximum node does not exist, find the node in the set which has the minimum informational value, lower than the informational value of any of the bounding anchor nodes. If such a minimum node exists, mark it as an anchor node and connect it with a branch to each of the other anchor nodes. If neither a maximum nor a minimum node is found among the interpolant nodes of the given set, then the set of nodes represents a single "atomic" set (or network). Repeat this partitioning process

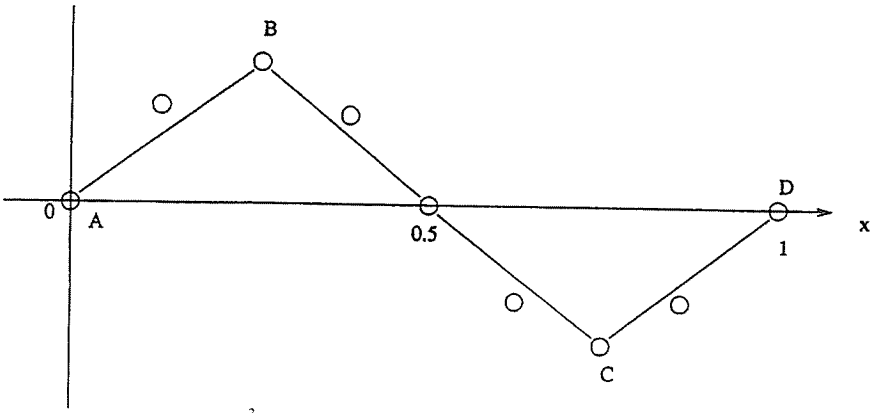

FIGURE 3. Schema of a D^3 network

separately for each of the sub-networks, until the original set has been partitioned into a set of "atomic" sub-networks, each of which is bounded by anchor nodes.

DINER^2: A SIMPLE EXAMPLE

To demonstrate DINER^{n+1}, consider the simple $D_N{}^2$ schematized in Figure 3. It is a period of a discretized sine wave. Imagine that it represents an experimental observation of some system information, which was obtained in that discrete form. Further, imagine that the observer did not have a priori knowledge of the functional form of the data distribution. If, say, one hundred (100) data nodes were collected, the problem here would be to find the smallest network $C_M{}^2$, ($M < 100$), which captures the information carried by the original $D_{100}{}^2$ network.

Of course, if the observer could visualize the data, he/she might immediately be able to fit a sine wave function to the distribution. The result would be $C_2{}^2$, a two-node network with one partition defined by the interpolation function: $d_j = a.\sin(2\pi x_j)$, $0 \leq x_j \leq 1$, and anchored at the boundary nodes A and D. That would correspond to ideal contraction. However, an observer of a system usually may not be able to completely visualize the discrete data representation of the system, or perhaps may not intuitively be able to recognize a single functional form with which to model such data. A scheme such as DINER^{n+1} must then be used.

In this simple example there are only two boundary nodes, A and D. Automatically they become the source and sink nodes, respectively, as well as the boundary anchor nodes. Applying the LBM anchoring algorithm to the set of boundary nodes A, D and the interpolant nodes of the data set, the node, B, with the highest data value is identified as the next anchor node. Branches, AB and BD are drawn to connect B to A and to D, respectively. The interpolant nodes between A and B form a sub-network, and those between B and D form a second sub-network; thus, the anchor nodes A, B and D have partitioned $D_{100}{}^2$ into two sub-networks. Further application of the LBM anchoring algorithm to the sub-network bounded by A and B, yields no other sub-network, so the AB sub-network is atomic; but, on the BD sub-network, the node C is identified as a minimum node, since its data value is smaller than that of any other node in its sub-network. C is therefore an

anchor node, which partitions the BD sub-network into two sub-networks, each of which is seen to be atomic.

The original network D_{100}^2, has now been partitioned into three atomic sub-networks. AB, BC and CD bounded by the four anchor nodes, A, B, C and D. Each of the three partitions is defined by its two bounding anchor nodes and the same partition interpolation functional:$(L_j^2 + E_j^2)$.

Let: $z_j = (x_j - x_h)/(x_t - x_h)$ represent the normalized "distance" between the jth interpolant node in a partition and the partition's source node x-coordinate x_h, and let d_h and d_t be the data values at the partition's source and sink nodes, respectively. The components of the partition interpolation functional are:

$$L_j^2 = d_h + (d_t - d_h).z_j \tag{4}$$

$$E_j^2 = a.\sin(\pi/2)z_j^{\,b}\cos(\pi/2)z_j^{\,c} \tag{5}$$

The form of E_j^2 chosen here is by no means unique. Other, and perhaps more general, fitting methods, (Frank, 1987), may be exploited. But once chosen, the form of E_j^2 applies to any atomic partition of any D_N^2 network, with only the values of the partition interpolation function coefficient array elements (a,b,c), differing among the partitions. The contraction ratio for a $DINER^2$ scheme which minimizes a D_N^2 to a C_M^2, using the interpolation functional forms suggested above, is:

$$\Gamma = 2N/(5M-3) \tag{6}$$

Since the original D_{100}^2 network was contracted to C_4^2, a network composed of the four anchor nodes, A, B, C and D, which define three partitions, each of which is represented by two bounding anchor nodes and an array of three numeric coefficients, the realized contraction ratio is approximately $\Gamma = 12$. The ideal contraction C_2^2, for the same network would have yielded a contraction ratio: $\Gamma = 40$.

CONCLUDING REMARKS

$DINER^{n+1}$ is a simple, generally consistent and easily programmable scheme for contracting digital information networks, with minimal loss. It is applicable to simple as well complex systems in a wide variety of contexts and dimensionality. Because the functional forms needed to represent a sub-network (or partition) of a D_N^{n+1} may be analytical, fractal or any other type, $DINER^{n+1}$ can draw from the strengths of any available knowledge base on discrete data fitting and functional analysis.

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