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Biased Random Walks

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The work to follow grew out of an attempt to generalize a standard random walk problem. The orderly presentation of our mathematics does not represent the process of developing the ideas. That process was actually quite haphazard as befits the subject.

A Standard Problem.

Let us suppose that a random walk takes place on the linear array of points $x = 1, 2, 3, ..., N$. On each "go" from an interior point, $x = 2, 3, 4, ..., N - 1$, the random walker must take a unit step either to his right or to his left. The probability of a move in either direction in 1/2.

$$x = 1\,2\,3\,\ldots\,N\text{-}1\,N$$

FIGURE 1. The Linear Array

Let p(x) represent the probability of reaching the endpoint $x = N$ from $x = 1, 2, 3, ..., N - 1, N$. Both $x = 1$ and $x = N$ serve as traps from which there is no escape. Thus $p(1) = 0$ and $p(N) = 1$. The probabilities must satisfy the average value condition

$$p(x) = (1/2)p(x - 1) + (1/2)p(x + 1) \qquad (1)$$

on the interior of the array. This equation implies that $p(x) = x/N$, and further computations show that p(x) is the unique function satisfying both Equation 1 and the boundary condition $p(1) = 0$, $p(N) = 1$. Finding p(x) is an example of a one-dimensional Dirichlet problem (Doyle and Snell, 1984; Boyd and Raychowdhury, 1989; Boyd and Raychowdhury, 1992a; Boyd and Raychowdhury, 1992b).

An Extension of the Problem.

We asked the question: "Are there functions f(x) such that Equation 1 when amended to

$$p(x) = \frac{1}{2}\left(\frac{f(x-1)}{f(x)}\right) p(x-1) + \frac{1}{2}\left(\frac{f(x+1)}{f(x)}\right) p(x+1) \qquad (2)$$

will uniquely define a probability of reaching $x = N$ from an interior point of the array?"

Restrictions upon the functions are that $f(x) \neq 0$, $f(x-1)/f(x)$ and $f(x+1)/f(x)$ are positive, and

$$\frac{1}{2}\left(\frac{f(x-1)}{f(x)}\right) + \frac{1}{2}\left(\frac{f(x+1)}{f(x)}\right) = 1. \qquad (3)$$

Equation 3 is necessary since the two ratios of functions represent the probabilities of moving to the left and right from the point with coordinate x. This equation can be rewritten as

$$(1/2)f(x-1) + (1/2)f(x+1) = f(x)$$

which repeats the average value property of Equation 1. Therefore, any such function $f(x)$ must be linear.

A Solution of the Extended Problem.

Equation 2 can be rewritten as

$$[f(x+1)\,p(x+1) - f(x)\,p(x)] - [f(x)\,p(x) - f(x-1)\,p(x-1)] = 0$$

which suggests that the second of the successive differences in the product function $f(x)p(x)$ for unit increments in x is zero. Therefore, we are led to consider $f(x)p(x) = Ax + B$ where the constants A and B need to be determined from the choice of $f(x)$ and the boundary conditions on $p(x)$.

For example, suppose that $f(x) = x$, $p(1) = 0$, and $p(N) = 1$ so that $p(x) = (Ax + B)/x$. Then $p(1) = 0$ implies that, $A = -B$ and $p(N) = 1$ implies that $A = N/(N-1)$. Thus $p(x) = (N/(N-1))((x-1)/x)$ and further computation shows that $f(x)$ satisfies Equation 3 while $p(x)$ satisfies Equation 2.

Equations 2 and 3 also imply that the maximum and minimum values of $p(x)$ must occur at the endpoints of the walk ($x = 1$ and $x = N$). Therefore, two probability functions $p(x)$ and $p'(x)$ which agree at $x = 1$ and $x = N$ must necessarily be identical on all of $\{1, 2, 3, ..., N\}$.

Example 1.

We now give a particular example to indicate the sort of computations that can be made. Let $N=4$ and $f(x) = x$ so that $p(x) = \frac{4}{3}\left(\frac{x-1}{x}\right)$. Then $p(1)=0$, $p(2) = \frac{2}{3}$, $p(3) = \frac{8}{9}$ and $p(4) = 1$.

We can also construct the 4-by-4 transition matrix $(p(i,j))$ in which $p(i,j)$ represents the probability of a move from $x = i$ to $x = j$ on a single step (Kemeny et al., 1965). The definition of our random walk implies the following:

$$P(i,j) = 0 \text{ if } |i-j| > 1,$$

$$p(1,1) = 1 \text{ and } p(4,4) = 1,$$

$$p(1,2) = p(4,3) = 0, \text{ and}$$

$$p(2,1) = \frac{f(1)}{2f(2)},\ p(2,3) = \frac{f(3)}{2f(2)},\ p(3,2) = \frac{f(2)}{2f(3)},\ p(3,4) = \frac{f(4)}{2f(3)}.$$

Therefore,

$$\left(p(i,j)\right) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{4} & 0 & \frac{3}{4} & 0 \\ 0 & \frac{1}{3} & 0 & \frac{2}{3} \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The matrix $(q(i,j)) = (p(i,j))^n$ gives the probabilities of moving from $x = i$ to $x = j$ in exactly $n = 1, 2, 3, \ldots$ steps. That is, $q(i, j)$ is the probability of reaching $x = j$ from $x = i$ in n steps. Readers can verify for themselves that the entries of

$$\left(p(i,j)\right)^2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{4} & \frac{1}{4} & 0 & \frac{1}{2} \\ \frac{1}{12} & 0 & \frac{1}{4} & \frac{2}{3} \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

are as claimed.

Since we have computed the probabilities of reaching $x = 4$ for all walks taken to their conclusion and since $p(1) = 1 - p(4)$, we can give the nonobvious limit for $(p(i,j))^n$ as n tends to infinity:

$$\lim_{n \to \infty} \left(p(i,j)\right)^n = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{3} & 0 & 0 & \frac{2}{3} \\ \frac{1}{9} & 0 & 0 & \frac{8}{9} \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

This limit can be verified by the computation

$$\left(p\,(i,j)\right) \cdot \begin{pmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{3} & 0 & 0 & \frac{2}{3} \\ \frac{1}{9} & 0 & 0 & \frac{8}{9} \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{3} & 0 & 0 & \frac{2}{3} \\ \frac{1}{9} & 0 & 0 & \frac{8}{9} \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

A Two Dimensional Random Walk.

Let us now consider a random walk on the square array {1, 2, 3, ..., N} x {1, 2, 3, ..., N}. Suppose that the sides of the square are traps. Once the walk reaches a side, the walk becomes one dimensional with traps at the vertices (1, 1), (1, N), (N, N), and (N, 1). We are interested in reaching (N, N) from (x, y). We denote the probability of that event by $p(x, y)$.

On each side, the walk is prescribed by Equation 2. At the interior lattice point (x, y) which has the four nearest neighbors $(x - 1, y)$, $(x + 1, y)$, $(x, y - 1)$, and $(x, y + 1)$, we have the average value requirement

$$p\,(x, y) = \frac{1}{4}\left[\frac{f\,(x-1, y)}{f(x, y)}\, p\,(x-1, y) + \frac{f\,(x+1, y)}{f\,(x, y)}\, p\,(x+1, y) + \frac{f\,(x, y-1)}{f\,(x, y)}\, p\,(x, y-1) + \frac{f\,(x, y+1)}{f\,(x, y)}\, p\,(x, y+1)\right]$$

to replace Equation 2.

Straightforward computations show that, for $p(1, 1) = p(1, N) = p(N, 1) = 0$, $p(N, N) = 1$ and $f(x) = x$, $f(y) = y$, $f(x, y) = f(x)f(y)$, the function

$$p\,(x, y) = \frac{N^2}{(N-1)^2}\left(\frac{x-1}{x}\right)\left(\frac{y-1}{y}\right)$$

uniquely satisfies the problem and analogies with the one dimensional case hold as expected. The problem and its product solutions can be generalized to higher dimensions.

Example 2.

If we let N = 4, then $p(x, y) = \frac{16}{9}\left(\frac{x-1}{x}\right)\left(\frac{y-1}{y}\right)$. We display the values of $p(x, y)$ in the table below.

y \ x	1	2	3	4
1	0	0	0	0
2	0	$\frac{4}{9}$	$\frac{16}{27}$	$\frac{2}{3}$
3	0	$\frac{16}{27}$	$\frac{64}{81}$	$\frac{8}{9}$
4	0	$\frac{2}{3}$	$\frac{8}{9}$	1

TABLE 1. Values of $p(x, y)$ for $N = 4$.

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