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Napoleon's Theorem, Shakespeare's Theorem, and Desargues's Theorem

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ABSTRACT

A very short and inevitably successful proof of Napoleon's theorem is given in terms of complex algebra. Another two related theorems, also concerned with centroids, are then proved by using an even simpler vector method. I eponymize these theorems to Shakespeare and Bacon in order to obey Stigler's law of eponymy. Finally, Desargues's theorem is related to the 3D monocular effect.

My aim in this brief note is to entertain and perchance to educate rather than to produce something new under the sun.

Consider a triangle ABC, labelled anticlockwise in the order A, B, C. (See Fig. 1.) Construct three equilateral triangles CBP_1, ACP_2, and BAP_3 external to ΔABC. Then the centroids G_1, G_2, G_3 of these three equilateral triangles form another equilateral triangle which I shall call the *derived* Napoleonic triangle. Someone who respected the Godfather, named the result "Napoleon's theorem" and therefore, by Stigler's Law of Eponymy (Stigler, 1980, "eponymy is always wrong", see also Good, 1985), we infer that it was more likely to have been discovered by Henry VIII. One proof was given in this journal recently by Boyd and Raychowdhury (1992).

It is intuitively obvious in advance that a short and automatic proof can be obtained by the machinery of complex algebra (if no mistakes are made!). This proof might not be too familiar though I would be surprised if it hasn't been discovered many times in the last hundred years so I give it with no claim to originality. I describe the "machinery" in three sentences, followed by the proof in four sentences.

A complex number z can be used to denote either a point in the Argand diagram (complex plane) or the vector $\vec{Oz}$ from the origin to that point, or, by the definition of a vector, any vector obtained from $\vec{Oz}$ by parallel displacement. Thus $\vec{AB}$ can be written as $B - A$ (where B and A are complex numbers). For present purposes, O can be anywhere except "at infinity".

The line segment $\vec{CP}_1$, can be obtained by rotating the segment $\vec{AB}$ about C through an angle $\pi/3$ (anticlockwise); therefore $P_1 = C + \omega(B - C)$ where $\omega = \exp(\pi i/3)$. Therefore

$$G_1 = \frac{1}{3}[C + B + C + \omega(B - C)] = \frac{1}{3}[B(1 + \omega) + C(2 - \omega)]$$

and by "symmetry",

$$G_2 = \frac{1}{3}[C(1 + \omega) + A(2 - \omega)].$$

Therefore

$$\vec{G_1G_2} = G_2 - G_1 = \frac{1}{3}[A(2-\omega) - B(1+\omega) + C(2\omega - 1)].$$

Similarly,

$$\vec{G_2G_3} = \frac{1}{3}[A(2\omega - 1) + B(2 - \omega) - C(1 + \omega)$$

and this is seen to be $\omega^2\ \vec{G_1G_2}$ (an anticlockwise rotation of $\vec{G_1G_2}$ by $2\pi/3$) by using the equations $\omega^3 = -1$ and $\omega^2 = \omega - 1$.

Napoleon's second theorem (for example, Coxeter and Greitzer, p. 64) states that a similar result is true if the original equilateral triangles are all reflected in their "bases" BC, CA and AB. (The "inner" Napoleon theorem.) This can be proved by essentially the same argument as above, as the reader can verify. (Just change the sign of ω.)

Coxeter and Greitzer (1967, p. 166) prove that the outer and inner Napoleon derived triangles have the same centroid. This again follows "automatically" from the "complex" method because

$$\frac{1}{3}(G_1 + G_2 + G_3) = \frac{1}{9}\{[B(1+\omega) + C(2-\omega)] + [C(1+\omega) + A(2-\omega)] + [A(1+\omega) + B(2-\omega)]\} = \frac{1}{3}(A + B + C)$$

so the centroid of the outer derived Napoleonic triangle is at the centroid of the original triangle ABC, and similarly so is the centroid of the inner derived Napoloeonic triangle.

For a short history of Napoleon's theorem and allied matters, with many references, see Wetzel (1992).

Sometimes the most elementary use of vectors, without even appealing to inner products or vector products, is more powerful than the use of complex numbers. This is because vectors are defined in any number of dimensions. (For the advanced use of vectors, in fact Grassmanians, in geometry, see Forder, 1941.) This (elementary) method is illustrated by the proof of the following result which I shall call "Shakespeare's theorem" so as to bridge the "two cultures" (the humanities and the sciences) while continuing to obey Stigler's law. It could also be called a shrinkage theorem.

Given n points $A_1, A_2, \ldots, A_n$ in d dimensions ($d = 1, 2, 3, \ldots$), let G_i denote the centroid of the ($n-1$) points obtained by omitting A_i. Then the construct consisting of $G_1, G_2, \ldots, G_n$ is similar (in the standard technical sense of Euclidean geometry) to that consisting of $A_1, A_2, \ldots, A_n$ and is scaled down by a factor n-1. The two constructs have the same centroid. Moreover, corresponding sides are "antiparallel", that is, parallel but oriented in opposite directions. The theorem is illustrated in Figure 1 for the case $n = 4$, $d = 2$. (The case $n = 3$ is familiar and the case $n = 2$ is trivial.) To visualize Figure 2 as a theorem about a tetrahedron ($n = 4$, $d = 3$) instead of a quadrilateral, it is better viewed with only one eye. That eye acts as a center of projection. With both eyes open one has more visual cues (conscious or subconscious) that a diagram is in the plane of the paper.

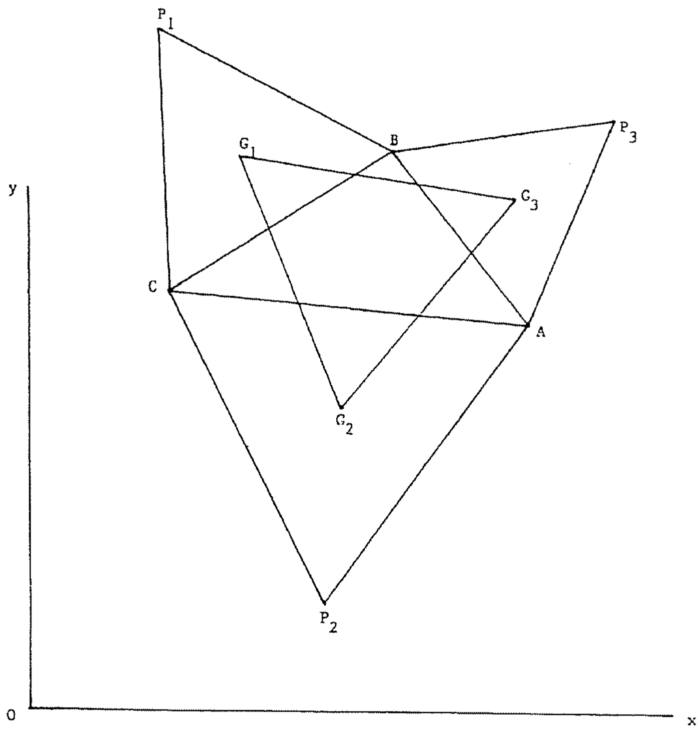

FIGURE 1. Napoleon's theorem.

Proof of Shakespeare's theorem. We have

$$G_1 = \frac{1}{n-1}(A_2 + A_3 + \ldots + A_n)$$

and

$$G_2 = \frac{1}{n-1}(A_1 + A_3 + A_4 + \ldots + A_n)$$

so

$$\overrightarrow{G_1G_2} = G_2 - G_1 = \frac{1}{n-1}(A_1 - A_2)$$

$$= \frac{1}{n-1}\,\overrightarrow{A_2A_1}$$

and similarly

$$\overrightarrow{G_iG_j} = \frac{1}{n-1}\,\overrightarrow{A_jA_i}$$

for all pairs i and j. This equation expresses the required "antiparallelism" and a scaling down of the original construct by a factor (n-1). The main part of the

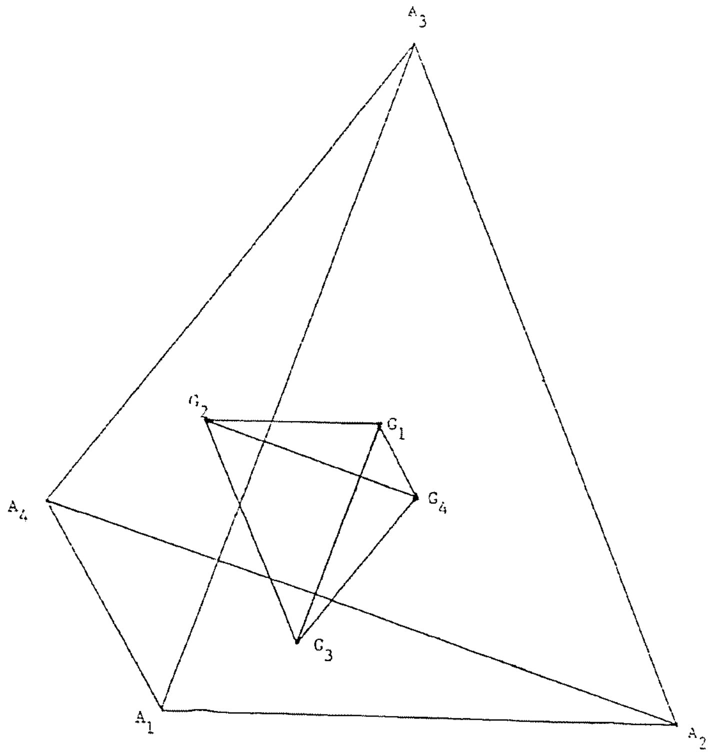

FIGURE 2. The shrunken quadrilateral or tetrahedron. If the theorem were used repeatedly, the successive tetrahedra would shrink down to a point, namely G.

theorem now follows from the fact that two figures (each connected and without "hinges") are congruent if one can be obtained from the other by a length-preserving transformation (see, for example, Coxeter and Greitzer, 1967, p. 80).

The proof that the two constructs have the same centroid is too simple to include here.

Bacon's Theorem. The lines $\overrightarrow{A_iG_i}$ ($i = 1, 2, ..., n$) *are concurrent.*

Proof. Denote by G the centroid of $A_1, A_2, ..., A_n$. We have

$$G = (A_1 + A_2 + ... + A_n)/n$$
$$= (1/n) A1 + (1 - 1/n)G_1 .$$

Because the sum of the two coefficients is 1, it follows that G lies on the line $\overrightarrow{A_1G_1}$ and similarly on the lines $\overrightarrow{A_iG_i}$ for all i. *A fortiori* these n lines are concurrent and we have brought home the Bacon.

Excuse for the eponymy. It has often been conjectured that "Shakespeare" was a pseudonym for Francis Bacon.

Statistical interpretation. Figure 2 and the allied discussion have an obvious interpretation (in a multivariate way) in terms of the Introduction of Efron (1982). The similarity (in the technical sense) of the shrunken structure to the original one makes it intuitively obvious that the Jackknife technique of "missing one out" cannot lead to an improved estimate of the population mean. But, as Efron's Introduction implies, the Jackknife (and Bootstrap) were not proposed in relation to the population mean.

"Projecting back" to one eye. Desargues's famous theorem in projective geometry states that if two triangles are in perspective from a point then they are also "in perspective from a line" (see, for example, Coxeter and Greitzer, 1967, p. 70). The theorem is thus expressible in terms of the incidence relations in only two dimensions, but its proof is not. When the axioms of incidence in three dimensions are assumed, the proof is simple because the two planes of the triangles meet in a line. See, for example, Baker (1943), pp. 22-23. This proof can be made especially intuitive if the diagram is viewed from one eye and is projected backwards into three dimensions. This visual technique of backward projection was used above to transform a quadrilateral into a tetrahedron. For another example, especially elegant, of viewing a two-dimensional diagram as if it were in three dimensions so as to give greater insight, see Honsberger (1976, page 19) who acknowledges Frank Bernhart.

The technique provides a bridge between geometry and artistic appreciation, as may very well have been noticed by Desargues (1591-1661) who was an architect as well as a geometer. For when a picture has strong perspective features, such as a road receding into the distance, it can be made to look especially three-dimensional when viewed with one eye closed. This is found to be true by a high fraction of male viewers. For a write-up of a sampling experiment on this topic see Good (1986). This "3D monocular effect" was noticeable by 28 out of the 37 subjects questioned, mostly male. Professor Dr. Anton Hajos of the University of Giessen, who is an expert on human visual perception, informed me in 1987 (i) that the 3D monocular effect is known, (ii) that it is less clear for photographs taken at a great distance, and (iii) that it is less noticeable by female subjects. He did not confirm that the effect is strongest for pictures containing a "lot of perspective". He cited Hofmann (1925, page 434 ff).

Only recently did I notice the relationship between this property of visual perception and the visual proof of Desargues's theorem although I became aware of the latter in about 1936 while attending a course on projective geometry given by F.P. White in Cambridge, England. My excuse for taking more than fifty years to notice the relationship is that unfortunately one tends to keep the two cultures in watertight compartments.

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