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Napoleon's Triangle

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ABSTRACT

Affine, convex coordinates were developed by the German mathematician A.F. Möbius (1790-1868). In this paper, we continue our work with convex coordinates and the Fermat point to show that Napoleon's (outer) triangle is equilateral.

INTRODUCTION

There is a story that J. J. Sylvester was once shown a mathematical statement and asked whether he thought it was true or not. After he replied that there was no way for it to be true, he was shown a paper of many years before. In that paper, Sylvester himself had proved the statement to be true.

We compare ourselves with Sylvester in only one respect–his forgetting something he once understood. In a newsletter which we received not too long ago, the following elegant geometric fact was presented:

Suppose that, on the sides of any triangle ($\Delta V_1V_2V_3$), equilateral triangles have been constructed so that the three equilateral triangles lie in the exterior of the original triangle. Then the triangle which has the centroids of the three equilateral triangles as its vertices is also equilateral.

CONVEX COORDINATES AND NAPOLEON'S TRIANGLE

The geometry of the triangles is sketched in Figure 1 where Q_1,Q_2,Q_3 are the centroids of the equilateral triangles upon the sides of $\Delta V_1V_2V_3$. The geometric fact, as we later learned, is quite well known with $\Delta Q_1Q_2Q_3$ dignified by the Emperor Napoleon's own name (Coxeter and Greitzer). However, none of this information accompanied the bald statement which only had to do with the thirty degree angles at V_1, V_2, , and V_3.

In Figure 1, θ_i is the angle at V_i, ℓ_i is the length of the side opposite V_i, P_i is the vertex of the equilateral triangle opposite V_i, and Q_i is the centroid of that equilateral triangle. When we first tried to prove this fact for ourselves, we did not see an obvious way to start. What we had forgotten was that the equilateral triangles are the same triangles needed for defining the Fermat point. (We thank Professor Ray Spaulding of Radford University for jogging our memory.)

We discussed the convex coordinates of the Fermat point of a triangle in this journal (Boyd, Lees, and Raychowdhury, 1990), and in this note we show how to use convex coordinates to prove that Napoleon's triangle is equilateral.

The Fermat point of $\Delta V_1V_2V_3$ is the point at which the lines $\overleftrightarrow{P_iV_i}$ are concurrent. We do not clutter our figure with these lines since they are not needed in the arguments to follow.

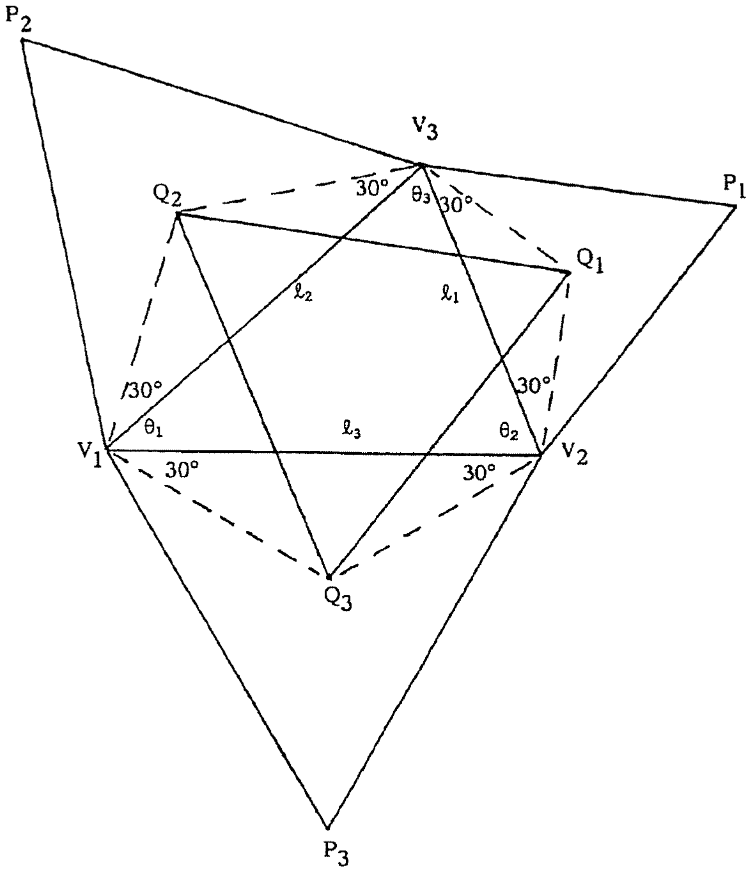

FIGURE 1. Napoleon's Triangle $-\Delta Q_1Q_2Q_3$.

However, in finding the convex coordinates of the Fermat point, we did find the convex coordinates of P_2, and we begin with that result. The convex coordinates of P_2 are

$$(\alpha_{12}, \alpha_{22}, \alpha_{32}) = \left(\frac{\ell_1 \ell_2 \sqrt{3} \cos\theta_3 + 2A}{4A}, \frac{\ell_2^2 \sqrt{3}}{4A}, \frac{-\ell_3 \ell_2 \sqrt{3} \cos\theta_1 + 2A}{4A} \right)$$

where α_{i2} is the convex coordinate of P_2 with respect to V_i and A is the area of $\Delta V_1V_2V_3$.

The affine nature of convex coordinates forces a symmetry permitting us to write the convex coordinates of P_1 and P_3 as well. The convex coordinates of P_1 are

$$(\alpha_{11}, \alpha_{21}, \alpha_{31}) = \left(\frac{-\ell_1^2 \sqrt{3}}{4A}, \frac{\ell_1 \ell_2 \sqrt{3} \cos\theta_3 + 2A}{4A}, \frac{\ell_1 \ell_3 \sqrt{3} \cos\theta_2 + 2A}{4A} \right)$$

and the convex coordinates of P_3 are

$$(\alpha_{13}, \alpha_{23}, \alpha_{33}) = \left(\frac{\ell_1 \ell_3 \sqrt{3} \cos\theta_2 + 2A}{4A}, \frac{\ell_2 \ell_3 \sqrt{3} \cos\theta_1 + 2A}{4A}, \frac{-\ell_3^2 \sqrt{3}}{4A} \right)$$

Without loss of generality, we can take the Cartesian coordinates of V_1, V_2, V_3 to be (0, 0), (1, 0), (a, b), respectively, where a,b are both positive. Then the relations

$$\ell_1^2 = (1-a)^2 + b^2,\ \ell_2^2 = a^2 + b^2,\ \ell_3^2 = 1,\ \ell_1\ell_2 \cos\theta_3 = (\ell_1^2 + \ell_2^2 - \ell_3^2)/2,$$
$$\ell_2\ell_3 \cos\theta_1 = (\ell_2^2 + \ell_3^2 - \ell_1^2)/2, \quad \ell_1\ell_3 \cos\theta_2 = (\ell_1^2 + \ell_3^2 - \ell_2^2)/2$$

and $A = b/2$ permit us to rewrite the convex coordinates in terms of just the two numbers a and b:

$$(\alpha_{11}, \alpha_{21}, \alpha_{31}) =$$
$$\left(\frac{-((1-a)^2 + b^2)\sqrt{3}}{2b}, \frac{(a^2 + b^2 - a)\sqrt{3} + b}{2b}, \frac{(1-a)\sqrt{3} + b}{2b} \right),$$

$$(\alpha_{12}, \alpha_{22}, \alpha_{32}) = \left(\frac{(a^2 + b^2 - a)\sqrt{3} + b}{2b}, \frac{-(a^2 + b^2)\sqrt{3}}{2b}, \frac{a\sqrt{3} + b}{2b} \right),$$

$$(\alpha_{13}, \alpha_{23}, \alpha_{33}) = \left(\frac{(1-a)\sqrt{3} + b}{2b}, \frac{a\sqrt{3} + b}{2b}, \frac{-\sqrt{3}}{2b} \right).$$

Now Q_i is the centroid of its triangle. Therefore, its convex coordinates with respect to the vertices of the triangle for which it is the centroid are $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. To obtain the convex coordinates of Q_i with respect to V_1, V_2, V_3 we simply use the "chain rule" for convex coordinates (Boyd and Raychowdhury, 1989).

The convex coordinates with respect to V_1, V_2, V_3 for Q_1 are

$$\begin{pmatrix} \alpha'_{11} \\ \alpha'_{21} \\ \alpha'_{31} \end{pmatrix} = \begin{pmatrix} \alpha_{11} & 0 & 0 \\ \alpha_{21} & 1 & 0 \\ \alpha_{31} & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{pmatrix} = \begin{pmatrix} \frac{-((1-a)^2 + b^2)\sqrt{3}}{6b} \\ \frac{(a^2 + b^2 - a)\sqrt{3} + 3b}{6b} \\ \frac{(1-a)\sqrt{3} + 3b}{6b} \end{pmatrix}.$$

The convex coordinates with respect to V_1, V_2, V_3 for Q_2 are

$$\begin{pmatrix} \alpha'_{12} \\ \alpha'_{22} \\ \alpha'_{32} \end{pmatrix} = \begin{pmatrix} 1 & \alpha_{12} & 0 \\ 0 & \alpha_{22} & 0 \\ 0 & \alpha_{32} & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{pmatrix} = \begin{pmatrix} \frac{(a^2 + b^2 - a)\sqrt{3} + 3b}{6b} \\ \frac{-(a^2 + b^2)\sqrt{3}}{6b} \\ \frac{a\sqrt{3} + 3b}{6b} \end{pmatrix}.$$

The convex coordinates with respect to V_1, V_2, V_3 for Q_3 are

$$\begin{pmatrix} \alpha'_{13} \\ \alpha'_{23} \\ \alpha'_{33} \end{pmatrix} = \begin{pmatrix} 1 & 0 & \alpha_{13} \\ 0 & 1 & \alpha_{23} \\ 0 & 0 & \alpha_{33} \end{pmatrix} \begin{pmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{pmatrix} = \begin{pmatrix} \frac{(1-a)\sqrt{3} + 3b}{6b} \\ \frac{a\sqrt{3} + 3b}{6b} \\ \frac{-\sqrt{3}}{6b} \end{pmatrix}.$$

Since the Cartesian coordinates of Q_i are $(0\alpha'_{1i} + \alpha'_{2i} + a\alpha'_{3i}, b\alpha'_{3i})$, we can write the Cartesian coordinates of the vertices of $\Delta Q_1 Q_2 Q_3$:

$$Q_1: \left(\frac{3a + b\sqrt{3} + 3}{6}, \frac{(1-a)\sqrt{3} + 3b}{6}\right),$$

$$Q_2: \left(\frac{3a - b\sqrt{3}}{6}, \frac{a\sqrt{3} + 3b}{6}\right), \text{ and}$$

$$Q_3: \left(\frac{1}{2}, -\frac{\sqrt{3}}{6}\right).$$

Note that the Cartesian coordinates of Q_3 are as expected. Finally, we compute the lengths of the sides of $\Delta\, Q_1 Q_2 Q_3$ by the Pythagorean Theorem to find that

$$Q_1Q_2 = Q_2Q_3 = Q_3Q_1 = \sqrt{\frac{a^2 + b^2 - a + b\sqrt{3} + 1}{3}}.$$

We have shown that Napoleon's triangle is equilateral by using convex coordinates and continuing the calculations begun in our discussion of the Fermat point.

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