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An Anharmonic Oscillator

James N. Boyd, St. Christopher's School,
Richmond, VA 23226

ABSTRACT

A uniform sphere bobbing in water furnishes an example of an anharmonic oscillator. The motion of the sphere is compared with the oscillation of a block of constant cross section which also floats in water. The period of the bobbing sphere is obtained as an elliptic integral. Projects and problems which follow from a study of the bobbing sphere are appropriate in applied mathematics and in intermediate level mechanics.

INTRODUCTION

Some time ago, our attention was caught by an exercise in numerical integration. The problem was to evaluate a definite integral for the period of oscillation of a sphere as it bobs while floating in water (Shampine and Allen). Since the integral formula was given without derivation, we decided to derive the formula for ourselves.

After we had finished, the derivation seemed at least as interesting as the original exercise. Therefore, after looking at the bobbing of a block of constant cross section, we shall give a derivation of the formula for the period of the bobbing sphere in the case that the density of the sphere is one half that of water.

THE BOBBING BLOCK

The bobbing in water of a wooden block of uniform density and constant horizontal cross section is almost simple harmonic. That is, if we ignore the viscosity of the water, the equation of motion of the block takes the form $\ddot{y} = -\omega^2 y$. We assume that the amplitude of vibration is not so large that the top of the block will ever be submerged.

Let A denote the area of a cross section of the block taken parallel to the surface of the water and $\rho (0 < \rho < 1)$ denote the specific gravity of the block. In Figure 1, we show the block with instantaneous water line at a vertical height y above the equilibrium level. The meaning of a positive value of y is that the block has been depressed from its floating rest position by a vertical distance y.

With Archimedes' Principle, we can find the buoyant force acting on the block. Then the unbalanced downward force (F) on the block as shown is its own weight minus the buoyant force which is just the weight of the displaced water. The volume of the block is Ah where h is height of the block. Since the density of water is 1 gm/cm^3 and the specific gravity of the block is ρ, the mass and weight of the block become ρAh and $g\rho Ah$, respectively. The volume and mass of the displaced water have the same numerical value, implying an upward force of $g\rho Ah + gAy$ due to buoyancy.

From Newton's Second Law, we can write the equation of motion for the block as

$$F = \rho Ah\ddot{y} = g\rho Ah - (g\rho Ah + gAy) = -gAy. \tag{1}$$

The term on the far right of the extended equation is negative because the net force is a restoring force in direction opposite to the displacement. An analysis of the

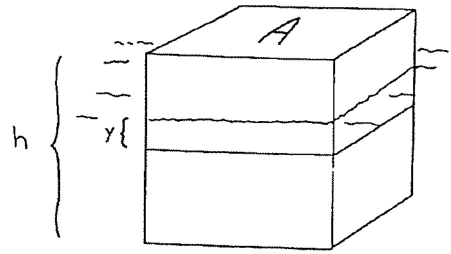

FIGURE 1. The bobbing block.

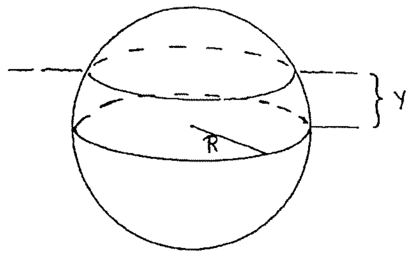

FIGURE 2. The bobbing sphere.

motion when the block has been raised with respect to the water level also yields a restoring force and the equation of motion is again given by 1. We now recognize this equation to be the standard equation of simple harmonic motion: $\ddot{y} = -\omega^2 y$ with $\omega^2 = g/(\rho h)$.

Rewriting this differential equation as $\dot{y}(d\,\dot{y}/d\,y) = -\omega^2 y$ permits us to integrate both sides to obtain

$$\dot{y}^2 = \omega^2 y_o^2 - \omega^2 y^2 \tag{2}$$

where y_o is the value of y when $\dot{y} = 0$. It is easy to see that $y = y_o \sin(\omega t + \delta)$ satisfies Equation 2. We have taken the trouble to give the equation because we will encounter an analogous equation when we derive the period of the bobbing sphere.

THE BOBBING SPHERE

Let us suppose that a sphere of radius R floats half submerged in water. That is, the specific gravity of the sphere is 0.5 and the center of the sphere is at the level of the water whenever the sphere floats in equilibrium. Let us next imagine that the sphere is pushed down until its center is a vertical distance q ($0 \leq q \leq R$) below the surface of the water. Then the sphere is released. In computing the period of the resulting motion, we shall ignore all effects due to friction and viscosity.

Unlike the block of constant cross section, the oscillating sphere does not displace a volume of water which is linearly proportional to its displacement. However, let us note that, for very small vertical displacements from equilibrium, the cross sections of the sphere at the water level do vary so slightly that the motion will approximate simple harmonic motion in the way the simple pendulum does for small angular displacements.

Consider the sphere when its center is at the depth y below the surface of the water. Then the volume of water displaced by the sphere (beyond its equilibrium displacement of $(2/3)\,\pi R^3$) is given by $\int_0^y \pi(R^2 - y^3/3)\,dy = \pi(R^2 y - y^3/3)$, and the weight of this displaced water is $g\,\pi(R^2 y - y^3/3)$ since the density of the water is

1 gm/cm^3. The unbalanced, buoyant force acting on the sphere must be this extra weight of displaced water. Then, since the force is a restoring one, Newton's Second Law gives us

$$(1/2)(4/3)\,\pi R^3\ddot{y} = -\,g\pi(R^2y - y^3/3) \tag{3}$$

where the coefficient of y is the mass of the sphere. The same equation of motion follows if the motion begins with the center of the sphere lifted a distance q above the water level. The equation of motion can be simplified to become

$$2R^3\,\ddot{y} = -\,g(3R^2y - y^3).$$

Noting that $\ddot{y} = \dot{y}(d\dot{y}/dy)$ we can antidifferentiate to obtain

$$R^3\dot{y}^2 = -\,g(3R^2y^2/2 - y^4/4) + g(3R^2q^2/2 - q^4/4) \tag{4}$$

since $y = q$ when $\dot{y} = 0$. This last equation is the analog of Equation 2 which we wrote for the bobbing block. After a bit of straightforward but tedious algebra, we can rewrite Equation 4 as

$$\left(2R\sqrt{\frac{R}{g(6R^2 - q^2)}}\right)\dot{y} = -\,q\sqrt{1 - \left(\frac{y}{q}\right)^2}\ \sqrt{1 - k^2\left(\frac{y}{q}\right)^2}$$

where $k^2 = q^2/(6R^2 - q^2)$. The negative sign on the righthand side of the equation was chosen since we intend to look at the motion of the sphere only during the first quarter of its period while $\dot{y}$ and y have opposite algebraic signs. Note that as $q \rightarrow 0$, $k \rightarrow 0$ also and we return to the simple harmonic approximation for the motion.

Letting $y/q = \sin s$ and $\dot{y} = dy/dt = q(\cos s)(ds/dt)$ and making the appropriate substitutions in the last equation, we can write

$$-\,dt = 2R\sqrt{\frac{R}{g(6R^2 - q^2)}}\ \cdot\ \frac{ds}{\sqrt{1 - k^2\sin^2 s}}\ . \tag{5}$$

If we now integrate both sides of Equation 5 over the first quarter period of motion (T/4), we obtain the desired integral formula for the period of oscillation

$$T = 8R\sqrt{\frac{R}{g(6R^2 - q^2)}}\ \cdot\ \int_0^{\frac{\pi}{2}} \frac{ds}{\sqrt{1 - k^2\sin^2 s}}$$

where k is as previously defined. The definite integral is a complete elliptic integral of the first kind with modulus k. Such integrals have been studied extensively.

OBSERVATIONS

Certainly neither the mathematics nor the physical arguments which we have presented are new. In fact, they are so "old fashioned" that they are quite likely to be unfamiliar and hence interesting to students today. The derivation of the frequency for the bobbing block can serve as an assignment for beginning physics students, and the derivation of the frequency for the sphere can serve as a project for more advanced students who have already thought about the simple pendulum swinging through large angular displacements. The problem for the sphere can be made more difficult by taking its specific gravity to be something other than 0.5. Then the equilibrium water level will not be at a horizontal equator of the sphere, and the bobbing of the sphere will no longer be symmetric with respect to the equator.

When divorced from our derivation, the evaluation of the definite integral for the period of our bobbing sphere furnishes a nice exercise in numerical integration and computer programming. Such an exercise motivated this note. We chose to give our derivation in the c.g.s. system of units because the unit density of water seemed to simplify our notation. However, once we have the formula before us, we can use any system we like provided that we give g, the acceleration due to gravity, in the appropriate units. For example, we used Simpson's Rule to find the period of oscillation for a sphere of radius one foot when q, the initial depth of the center, was taken to be the radius. We took g to have the value 32.174 ft/sec^2. We ran our program on an Apple IIe computer and obtained a period of 1.107 sec.

In doing the mathematics for the bobbing sphere there is a great deal to be learned about integration. Just as Equations 1 and 2 can be solved in closed form with the circular functions, Equations 3 and 4 can be solved in closed form with the elliptic functions. Equation 4 is satisfied by Legendre's sine amplitude function

$$y = \operatorname{sn}\left(\frac{\sqrt{g(6R^2 - q^2)}}{2R\sqrt{R}}\; t + \delta, k\right)$$

with domain taken to be the set of real numbers (Moulton). Although we still concede that the mathematics of our problem has been well understood for a long time, we suspect that it is not well known to students that every integral of the form

$$\int R\left(x, \sqrt{\sum_{i=0}^{4} a_i x^i}\right) dx,$$

where R is a rational function, can be expressed in terms of the elementary functions and elliptic integrals (Sokolnikoff and Redheffer).

Because of its many ramifications and possible extensions, we feel that the problem of the bobbing sphere provides a useful and rewarding project for the physics or applied mathematics class.

A FINAL NOTE

This paper (as regarded by its author) is an exercise in well established but now unfamiliar mathematics and not a rigorous derivation of an equation of motion. In the "good old days of closed form solutions," a distinguishing mark of a fine mathematician was his ability to recognize (before attempting) problems which were solvable and to make (within reason) whatever simplifying assumptions were necessary to produce manageable, yet still reasonable models of physical processes.

That is, he had a feel for the "simplest, meaningful" problem. The simplifying assumption about the bobbing sphere which leads to an "elegant" closed form solution is that the total force by the liquid on the sphere is the buoyant force of Archimedes' Principle. In actuality, that force would be the total force only in the case of static equilibrium. However, if the displacements and velocities of the sphere are taken to be small, the substitution of the buoyant force for the total force seems not unreasonable and is in the spirit of the "simplest, meaningful" problem. In addition, we ought to note that the computation for the period when the sphere is initially submerged stretches the model to an extent that it may become suspect to the physicist while remaining pleasing to the mathematician.

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