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The Convex Coordinates of the Fermat Point

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ABSTRACT

Convex coordinates are often called barycentric coordinates. They were introduced into mathematics by the German geometer A. F. Möbius (1790 - 1868), but their usefulness in computational geometry is not well known to mathematics teachers. In this paper, we briefly discuss convex coordinates and then find the convex coordinates of the Fermat point of a triangle

INTRODUCTION

We have taken our own good advice(Boyd and Raychowdhury, 1987b) and continued to seek ways to apply convex coordinates in Euclidean geometry. We obtained a copy of *Excursions in Advanced Euclidian Geometry* by A. S. Posamentier in hopes of finding a new problem for investigation. We were fortunate to find therein a discussion of the Fermat point of a triangle (Posamentier, 1984). As usual in "advanced Euclidean geometry," the mathematics is elegant, but the arguments demand a great deal of ingenuity. Most mathematicians and scientists not up on the intricacies of synthetic proof in Euclidean geometry would be hard put to derive the results in the manner described by Posamentier. However, with convex coordinates, many of the results can be obtained in straightforward, workmanlike fashion. Convex coordinates are tools for the blue collar geometer!

CONVEX COORDINATES

Before proceeding, let us remind the reader that the convex (or barycentric) coordinates of a point (P) in the plane can be taken as weights or forces which, when applied normal to the plane at the vertices of the triangle, cause the point to become the balance point for the three forces (Boyd, 1985). The three vertices of the triangle determine the plane of the triangle, and convex coordinates provide a natural means for investigating the concurrence of lines through these vertices.

Suppose that our triangle has vertices V_1, V_2, V_3 as shown in Figure 1. If forces α_i at V_i ($i = 1, 2, 3$) with $\alpha_1 + \alpha_2 + \alpha_3 = 1$ determine balance point P, then the convex coordinates of P are $(\alpha_1, \alpha_2, \alpha_3)$ with respect to the vertices in the order V_1, V_2, V_3. If and only if all three convex coordinates are nonnegative, P belongs to the closed triangular region which we shall denote by $\Delta V_1V_2V_3$. The convex coordinates of V_1, V_2, V_3 are (1,0,0), (0,1,0), (0,0,1), respectively; and along the side opposite vertex V_i, all points have the convex coordinate $\alpha_i = 0$. In the notation which we shall adopt, the side opposite V_i has length l_i and the angle at V_i is denoted by θ_i.

A second interpretation of convex coordinates which is consistent with that in terms of forces involves areas. Denoting the area of $\Delta V_1V_2V_3$ by A, we can write the convex coordinates of P as $(\alpha_1,\alpha_2,\alpha_3)$ = (Area $\Delta PV_2V_3/A$, Area $\Delta PV_3V_1/A$, Area $\Delta PV_1V_2/A$) (Boyd and Raychowdhury, 1987c).

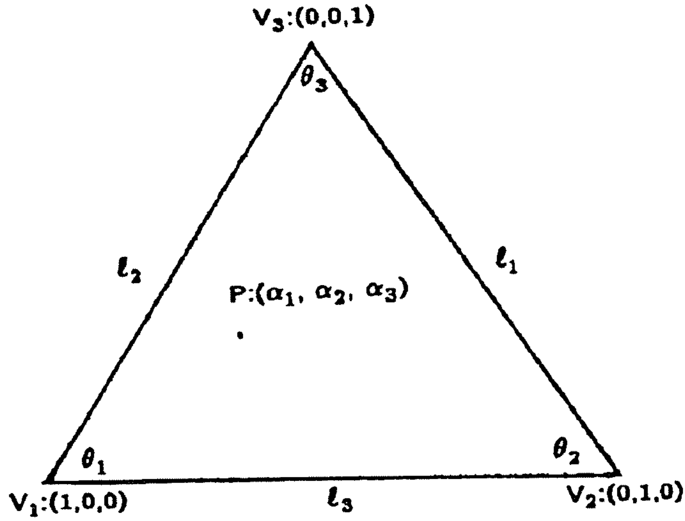

FIGURE 1. $\Delta V_1V_2V_3$ with Point P:$(\alpha_1, \alpha_2, \alpha_3)$.

Using forces and areas together, we can find the convex coordinates of the Fermat point of $\Delta V_1V_2V_3$. Posamentier defines the Fermat point and developes theorems about it as well as other special point of the triangle. However, we must lay a bit of groundwork before we can say what the Fermat point of $\Delta V_1V_2V_3$ is.

Upon each side of our triangle let us construct an equilateral triangle as shown in Figure 2. Of course, the equilateral triangle upon l_i has all sides of length l_i. The three vertices P_1, P_2, P_3 are in the exterior of $\Delta V_1V_2V_3$, and P_i is the vertex of the equilateral triangle constructed on l_i opposite to V_i.

We shall show that the three lines $\overleftrightarrow{P_1V_1}$, $\overleftrightarrow{P_2V_2}$, $\overleftrightarrow{P_3V_3}$ are concurrent at a point. That point is called the Fermat point (F) of the original triangle. Demonstrating the concurrence of the lines and finding the convex coordinates of the point are two forms of the same problem. The computations with the convex equations of lines in the plane are simiaar to those previously given for the circumcenter of a triangle (Boyd and Raychowdhury, 1987b).

THE CONVEX COORDINATES OF P_2

The first step in our task is to find the convex coordinates of one of the points P_1, P_2, P_3. We choose P_2. The area of $\Delta P_2V_3V_1$, upon l_2 is $l_2^2\frac{\sqrt{3}}{4}$. By the area interpretation of convex coordinates, the second coordinate for P_2 is

$$\alpha_2 = -\left(\frac{l_2^2\sqrt{3}}{4}\right)/A \qquad (1)$$

where A is the area of $\Delta V_1V_2V_3$ and the negative value of α_2 is due to the point's location outside of $\Delta V_1V_2V_3$ and in the interior of $\angle V_1V_2V_3$ (Boyd and Raychowdhury, 1987a).

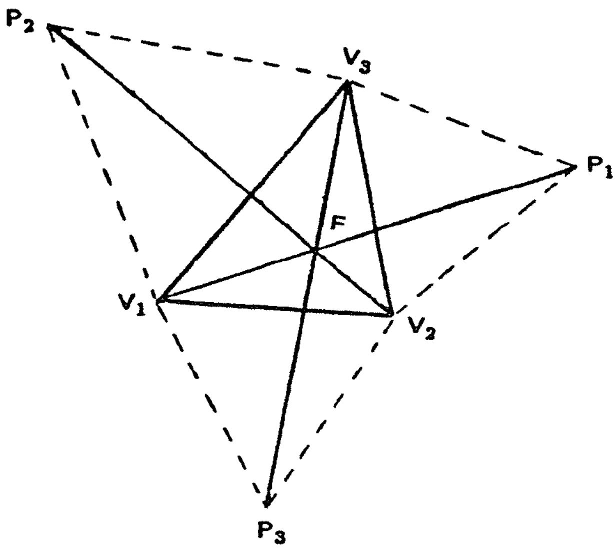

FIGURE 2. The Fermat Point (F) of $\Delta V_1V_2V_3$.

The equation of the perpendicular bisector of l_2 follows from the interpretation of convex coordinates as forces and the applocation of Archimedes' Law of Levers (Boyd and Raychowdhury, 1987a). If the perpendicular bisector of l_2 is to be a balance line for forces α_1, α_2, α_3 at V_1,V_2,V_3, respectively, we have the condition that

$$l_2\alpha_1 + (2\, l_1 \cos\theta_3 - l_2)\,\alpha_2 - l_2\,\alpha_3 = 0 \tag{2}$$

Since P_2 is a point of the line, its coordinates must satisfy equation 2.

Finally, we have

$$\alpha_1 + \alpha_2 + \alpha_3 = 1 \tag{3}$$

from the definition of convex coordinates.

Solving equations 1, 2, 3 simultaneously, we have

$$(\alpha_1\,,\alpha_2\,,\alpha_3) = \left(\frac{l_1\, l_2\sqrt{3}\cos\theta_3 + 2A}{4A}, -\frac{l_2^2\sqrt{3}}{4A}, \frac{l_3\, l_2\sqrt{3}\,\cos\theta_1 + 2A}{4A}\right).$$

The form of α_3 follows from the observation that $l_1 \cos\theta_3 = l_2 - l_3\cos\theta_1$.

THE CONVEX COORDINATES OF F

Along the line $\overleftrightarrow{P_2V_2}$, the convex coordinates α_1 and α_3 are in constant ratio (Boyd, Farley, and Raychowdhury, 1987). Thus the convex equation of $\overleftrightarrow{P_2V_2}$, is

$$(l_1\, l_2\sqrt{3}\,\cos\theta_3 + 2A)\,\alpha_3 = (l_3\, l_2\sqrt{3}\,\cos\theta_1 + 2A)\,\alpha_1. \tag{4}$$

Similarly, the convex equations of $\overleftrightarrow{P_1V_1}$ and $\overleftrightarrow{P_3V_3}$ are

$$(l_1\, l_3\sqrt{3}\,\cos\theta_2 + 2A)\,\alpha_2 = (l_1\, l_2\sqrt{3}\,\cos\theta_3 + 2A)\,\alpha_3 \tag{5}$$

and

$$(l_3 l_2 \sqrt{3} \cos\theta_1 + 2A)\alpha_1 = (l_1 l_3 \sqrt{3} \cos\theta_2 + 2A)\alpha_2 \qquad (6)$$

respectively.

In equations 4, 5, 6, let

$B_1 = l_3 l_2 \sqrt{3} \cos\theta_1 + 2A$,

$B_2 = l_1 l_3 \sqrt{3} \cos\theta_2 + 2A$, and

$B_3 = l_1 l_2 \sqrt{3} \cos\theta_3 + 2A$. The three equations become $B_3\alpha_3 = B_1\alpha_1$, $B_2\alpha_2 = B_3\alpha_3$, $B_1\alpha_1 = B_2\alpha_2$ which are satisfied by

$$\alpha_1 = \frac{B_2B_3}{B_1B_2 + B_1B_3 + B_2B_3},$$

$$\alpha_2 = \frac{B_1B_3}{B_1B_2 + B_1B_3 + B_2B_3}, \text{ and}$$

$$\alpha_3 = \frac{B_1B_2}{B_1B_2 + B_1B_3 + B_2B_3}.$$

Since these last three numbers sum to 1, they are indeed the convex coordinates of the point common to the lines $\overleftrightarrow{P_1V_1}$, $\overleftrightarrow{P_2V_2}$, and $\overleftrightarrow{P_3V_3}$. Thus we have established that the lines are concurrent and, at the same time, found the desired convex coordinates of the Fermat point. We do not bother to rewrite the rather pleasing expressions for the convex coordinates in terms of the l_i's and θ_i's.

FACTS CONCERNING THE FERMAT POINT

The Fermat point has several fascinating properties which can be developed by computation with the convex coordinates which we have found. Among these properties are that $\angle V_1FV_2 = \angle V_2FV_3 = \angle V_3FV_1 = 120^\circ$ and that F is the point of least total distance from the three vertices of the triangle (provided that each angle of the triangle is less than or equal to 120°), and that the lengths of $\overline{P_1V_1}, \overline{P_2V_2}, \overline{P_3V_3}$ are all the same. As the reader has no doubt surmised, it was the great matematician Fermat who, in the mid-Seventeenth Century, obtained these results.

It is through the introduction of convex coordinates into analytic geometry that industrious mathematicians can develope these properties as theorems for themselves through straightforward (if tedious) computation. If the Cartesian coordinates of the vertices V_i are (x_i, y_i) for $i = 1, 2, 3$, then the Cartesian coordinates of point $P:(\alpha_1, \alpha_2, \alpha_3)$ must be $(x,y) = \left(\sum_{i=1}^{3} \alpha_i x_i, \sum_{i=1}^{3} \alpha_i y_i\right)$. Thus the Cartesian coordinates of the Fermat point of any triangle can be obtained from its convex coordinates, and the properties in question can be deduced by the methods of anlytic geometry.

We really cannot recommend that the reader pursue the calculations. It is much easier and much more efficient to read elegant proofs by someone else that these claims are correct than to dig the proofs out for ourselves with convex coordinates. But, on the other hand, if we were stranded on a desert island without our copy of *Excursions in Advanced Euclidean Geometry*, we could do the job (although without

Professor Posamentier's elegance) with convex coordinates and analytic geometry. That knowledge gives us a sense of mathematical self-sufficiency.

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