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Monotone and Convex Quadratic Spline Interpolation

Maria H. Lam
Department of Computer Science
Hampton University, Hampton, VA 23668

ABSTRACT

A method for producing interpolants that preserve the monotonicity and convexity of discrete data is described. It utilizes the quadratic spline proposed by Larry Schumaker which was subsequently characterized by Ronald DeVore and Zheng Yan. The selection of first order derivatives at the given data points is essential to this spline. We generalize an observation made by DeVore and Yan and propose an improved method to select these derivatives. The resulting spline is completely local, efficient, and simple to implement.

Keywords: Discrete data, quadratic splines, shape-preserving interpolation.

INTRODUCTION

We are concerned with the numerical solution of the following interpolation problem: Given a set of data points $\{(x_i, y_i)\}$ $i = 1, \ldots, n$ where $x_1 < x_2 < \ldots < x_n$, find f such that

$$f(x_i) = y_i, \qquad i = 1, \ldots, n$$

and f preserves the monotonicity and/or convexity of the given data. That is: f is monotone increasing or decreasing in those intervals $I_i = [x_i, x_{i+1}]$ where the data is monotone increasing or decreasing. Similarly, f is convex or concave in I_i where the data exhibits such a trend.

Several methods to solve this problem have appeared. Some use geometric construction to obtain f as in McAllister & Roulier, 1981,1978 and McLaughlin, 1983, while others use piecewise cubic Hermite polynomials (Fritsch & Carlson, 1980 and Fritsch & Butland, 1984), quadratic splines (DeVore & Yan, 1986; Schumaker, 1983), some sort of splines under tension (Foley, 1988; Irvine, Marin & Smith, 1986) and other means. The purpose of this paper is to give a method for the selection of first order derivatives at the data points $\{(x_i, y_i)\}_{i=1,\ldots,n}$ that will lead to the generation of shape-preserving quadratic splines solving this problem.

A general method proposed by Schumaker (Schumaker, 1983) for solving this problem by quadratic spline is examined in the next section. His method relies on proper selection of first order derivatives at the given data points. He computed the derivative s_i at an interior data point $P_i(x_i,y_i)$ as a weighted average of two neighboring slopes connecting P_{i-1} to P_i and P_i to P_{i+1}. The weights are taken to be the lengths of the longest lines containing the chords in I_{i-1} and I_i. s_1 and s_n are computed as an average of a slope and s_2, a slope and s_{n-1} respectively. This approach produces an algorithm that is global in the sense that changing one data point will modify the shape of the curve from this point on. We compute the derivatives as a harmonic weighted average of the two slopes provided they are not zero. The weights are constant. Our method is local and less cumbersome. It is described in the section after the next. Results and discussions are given in the last two sections.

SHAPE-PRESERVING QUADRATIC SPLINES

McAllister and Roulier (McAllister & Roulier, 1981) introduced an algorithm to compute a shape-preserving quadratic spline in 1981. They used rather involved geometric constructions in their method. The properties of monotonicity and convexity of f are related to the first order derivative. If the derivatives $\{s_i\}_{i=1,...,n}$ at each data point can be estimated such that the monotonicity and the convexity of the given data remain intact, then the problem is reduced to finding f such that

$$f(x_i) = y_i, \qquad f'(x_i) = s_i, \quad i=1,...,n. \tag{1}$$

In general, f will be a piecewise function consisting of the collection $\{f_i\}_{i=1,...,n-1}$ of functions such that f_i is defined on I_i, and f_i and f_i' interpolate y_i, y_{i+1} and s_i, s_{i+1}. We choose f as a quadratic spline. The problem becomes a simple two-point Hermite interpolation problem: Find a quadratic polynomial f_i over I_i such that

$$f_i(x_k) = y_k, \qquad f_i'(x_k) = s_k, \quad k=i,\ i+1. \tag{2}$$

Schumaker proved that there exists a unique quadratic polynomial f_i solving this problem if and only if

$$\frac{s_i+s_{i+1}}{2} = \frac{y_{i+1}-y_i}{x_{i+1}-x_i} \tag{3}$$

and f_i is given as

$$f_i(x)=y_i+s_i(x-x_i)+\frac{(s_{i+1}-s_i)(x-x_i)^2}{2(x_{i+1}-x_i)}$$

When condition (3) fails, he showed that corresponding to every u in (x_i,x_{i+1}), there exists a unique quadratic spline over I_i having u as a knot that satisfies condition (2) . This spline is obtained by assigning a derivative s^* at u such that condition (3) is satisfied on both $[x_i,u]$ and $[u,x_{i+1}]$. Let y^* and s^* (which are dependent upon y_i, y_{i+1}, s_i, s_{i+1} and the location of u) be the functional value and the first order derivative at u. In particular,

$$(4)\quad f_i(x) = \begin{cases} A_1+B_1(x-x_i)+C_1(x-x_i)^2 & x_i<x<u, \\ A_2+B_2(x-u)+C_2(x-u)^2 & u\leq x<x_{i+1}, \end{cases}$$

with

$$A_1=y_i, \qquad B_1=s_i, \qquad C_1=(s^*-s_i)/2\alpha$$
$$A_2=A_1+B_1\alpha+C_1\alpha^2,\ B_2=s^*,\ C_2=(s_{i+1}-s^*)/2\beta$$

where

$$(5)\qquad s^*=f_i'(u)$$

$$=\frac{2(y_{i+1}-y_i)-(\alpha s_i+\beta s_{i+1})}{x_{i+1}-x_i},$$

$$\alpha = u-x_i$$

$$\beta = x_{i+1}-u.$$

If $s_i s_{i+1} \geq 0$, this spline is monotone on I_i if and only if $s_i s^* \geq 0$. If $s_i < s_{i+1}$, it is convex on I_i if and only if $s_i \leq s^* \leq s_{i+1}$. Similarly, if $s_i > s_{i+1}$, then it is concave if and only if $s_i \geq s^* \geq s_{i+1}$.

Hence this problem is solved provided that the s_i's and the u's can be chosen such that these conditions are met. Schumaker also formalized the selection of u that will lead to the preservation of the convexity of the data. The open question now lies on the selection of the derivatives, si, such that the monotone property can also be fulfilled.

As noted previously in the introduction, the computation used by Schumaker to obtain his derivatives has the potential difficulty of adjusting to changes in data points. The effect of changing a data point will propagate all the way to the end of the curve which may not be desirable. It is observed by DeVore and Yan that if si is defined as the harmonic mean of the two slopes and different weights are used to compute s_1 and s_n, Schumaker's spline becomes the quadratic spline proposed by McAllister and Roulier. The cost (in terms of the number of computations and data storage requirement) to compute the harmonic mean is low when compared to that of Schumaker's. Hence we choose to follow the harmonic mean approach and generalize it to provide a greater variety of shapes.

DERIVATIVE SELECTIONS

Our effort is focused around the selection of the derivatives s_i's. We wish to devise a method to calculate the s_i's that is easier to use than Schumaker's. This can be achieved if the chosen s_i's automatically fulfill some sufficient condition for monotonicity. The convexity/concavity of the data is preserved through Schumaker's knot insertion scheme.

As a result of the continuity conditions on the functional values and the first order derivatives of Schumaker's quadratic polynomial spline, for every u in the open internal (x_i, x_{i+1}), the corresponding s* and y* are given as follows:

Let

$$\delta_i = \frac{y_{i+1}-y_i}{x_{i+1}-x_i},$$

$$\alpha^* = \frac{u-x_i}{x_{i+1}-x_i},$$

$$\beta^* = 1-\alpha^*$$

$$= \frac{x_{i+1}-u}{x_{i+1}-x_i}.$$

Condition (5) can be expressed as

$$s^* = 2\frac{y_{i+1}-y_i}{x_{i+1}-x_i} - \frac{u-x_i}{x_{i+1}-x_i}s_i - \frac{x_{i+1}-u}{x_{i+1}-x_i}s_{i+1}$$

$$= 2\delta_i - \alpha^* s_i - \beta^* s_{i+1}$$

$$= 2\delta_i - \left(\alpha^* s_i + \beta^* s_{i+1}\right)$$

Condition (4) provides

$$y^* = y_i + s_i(x-x_i) + \frac{\left(s^*-s_i\right)(x-x_i)^2}{2(u-x_i)}.$$

Schumaker devised a criterion to select a u in the interval (x_i, x_{i+1}) such that the resulting quadratic spline is convex/concave as required. This u determines the y* and s* associated with it. It can be proved that if

(6) $$|s_i| \leq 2|\delta_i| \text{ and } |s_{i+1}| \leq 2|\delta_i|$$

then the spline is also monotone (see DeVore & Yan, 1986). This is a sufficient condition but not necessary for monotonicity.

For $i=2,\ldots,n-1$, if s_i is chosen as the harmonic mean of δ_i and δ_{i-1} when δ_i, δ_{i-1} are both positive or negative, *i.e.*

$$s_i = \begin{cases} \dfrac{\delta_{i-1}\delta_i}{0.5\delta_{i-1}+0.5\delta_i} & \text{if } \delta_{i-1}\delta_i>0 \\ 0 & \text{otherwise} \end{cases}$$

(7) $$s_1 = \begin{cases} 2\delta_1 - s_2 & \text{if } \delta_1(2\delta_1 - s_2)>0 \\ 0 & \text{otherwise} \end{cases}$$

(8) $$s_n = \begin{cases} 2\delta_{n-1} - s_{n-1} & \text{if } \delta_{n-1}(2\delta_{n-1} - s_{n-1})>0 \\ 0 & \text{otherwise} \end{cases}$$

where

$$\delta_{i-1} = \frac{y_i - y_{i-1}}{x_i - x_{i-1}},$$
$$\delta_i = \frac{y_{i+1} - y_i}{x_{i+1} - x_i}.$$

This automatically fulfills the sufficient condition (6) for monotonicity. Consequently the quadratic spline produced by these s_i's, together with Schumaker's knot insertion conserves the shape of the data.

Following along this line, we decide to try other convex combinations and set

$$s_i = \begin{cases} \dfrac{\delta_{i-1}\delta_i}{\xi\delta_{i-1}+\eta\delta_i} & \text{if } \delta_{i-1}\delta_i>0 \\ 0 & \text{otherwise} \end{cases}$$

where $0<\xi, \eta <1$ and $\xi + \eta = 1$. From this we derive a condition imposed upon ξ and η that will guarantee the fulfillment of condition (6) by s_{i-1} and s_i. This leads us to define s_i at an interior point (x_i,y_i) as

(9) $$s_i = \begin{cases} \dfrac{\delta_{i-1}\delta_i}{\xi\delta_{i-1}+\eta\delta_i} & \text{if } \delta_{i-1}\delta_i>0,\ (|\delta_{i-1}|-|\delta_i|)(\xi-0.5)\geq 0 \\ \dfrac{\delta_{i-1}\delta_i}{\eta\delta_{i-1}+\xi\delta_i} & \text{if } \delta_{i-1}\delta_i>0,\ (|\delta_{i-1}|-|\delta_i|)(\xi-0.5)< 0 \\ 0 & \text{otherwise} \end{cases}$$

s_1 and s_n are defined by (7) and (8).

For every fixed (ξ, η), the set of derivatives $\{s_i\}_{i=1,...,n}$ generated from (9) always satisfies the condition (6) because when $\delta_{i-1}\delta_i > 0$, $|\delta_{i-1}| \geq |\delta_i|$, and $\xi \geq 0.5$, we have

$$|s_i| = \frac{\delta_{i-1}\delta_i}{|\xi\delta_{i-1}+\eta\delta_i|}$$

$$= \frac{|\delta_i|}{\xi + \eta\frac{\delta_i}{\delta_{i-1}}}$$

$$< \frac{|\delta_i|}{\xi}$$

$$\leq 2|\delta_i|$$

and
$$|s_i| = \frac{|\delta_{i-1}|}{\xi\frac{\delta_{i-1}}{\delta_i} + \eta}$$

$$\leq \frac{|\delta_{i-1}|}{\xi+\eta}$$

$$= |\delta_{i-1}|$$

$$\leq 2|\delta_{i-1}|.$$

If $\delta_{i-1}\delta_i > 0$, $|\delta_{i-1}| \geq |\delta_i|$, and $\xi < 0.5$, then

$$|s_i| = \frac{|\delta_{i-1}\delta_i|}{|\eta\delta_{i-1}+\xi\delta_i|}$$

$$= \frac{|\delta_i|}{\eta+\xi\frac{\delta_i}{\delta_{i-1}}}$$

$$< \frac{|\delta_i|}{\eta}$$

$$\leq 2|\delta_i|$$

and

$$|s_i| = \frac{|\delta_{i-1}|}{\eta\frac{\delta_{i-1}}{\delta_i} + \xi}$$

$$\leq \frac{|\delta_{i-1}|}{\eta+\xi}$$

$$= |\delta_{i-1}|$$

$$\leq 2|\delta_{i-1}| .$$

The remaining cases are either obvious or can be proved similarly.

RESULTS AND EXAMPLES

When s_i is defined by (7), (8), (9) with $\xi = \eta = 0.5$ produces visually pleasing, shape-preserving quadratic splines proposed by McAllister and Roulier. Figures 1-2 display two such splines. The curves may show too much curvature around the areas where the convexity of the curve changes eventhough the shape of the data is still preserved. This is evident in Figure 2 between the 9th and the 10th data points. The midpoint of this interval is chosen as a knot. The curve changes from concave to convex at this knot with the monotonicity of the curve remaining unchanged.

Next, we experiment with different values for ξ, η. We apply them to the same data set and observe the change in the shape of the spline. As ξ and η deviate away from the value of 0.5, the curve becomes stiff. Figures 3 and 4 show the effect of different values of (ξ, η) being applied to the same data set used in Figure 2. Notice that the curve has tightened up between the 9th and the 10th data points resulting in a smoother transition from concave to convex. At the same time, the curve has tightened up between the 7th and the 8th points also producing a slightly "cornered" shape at the 8th point. One may use more than one set of (ξ, η) to a data set, applying one of them to an interval $[x_i, x_{i+1}]$ according to some criterion. Figure 5 shows the result of using two sets of (ξ, η). $\xi = \eta = 0.5$ is used throughout except at the 9th interval where the second divided difference changes sign, $\xi = 0.3$ and $\eta = 0.7$ values are used. Notice the improvement over Figure 2 between the 9th and 10th data points without sacrificing the smoothness between the 7th and 8th data points.

The two data sets used in Figures 1 and 2 are standard. The first one was introduced by Pruess and the second by Akima (see Pruess, 1978 and Akima, 1970).

Pruess' data:

	1	2	3	4	5	6	7	8	9	10	11
x	0	1	2	3	4	5	6	7	8	9	10
y	0	0.5	3.35	3.3	1.65	1.6	1.6	1.6	1.6	0.6	0

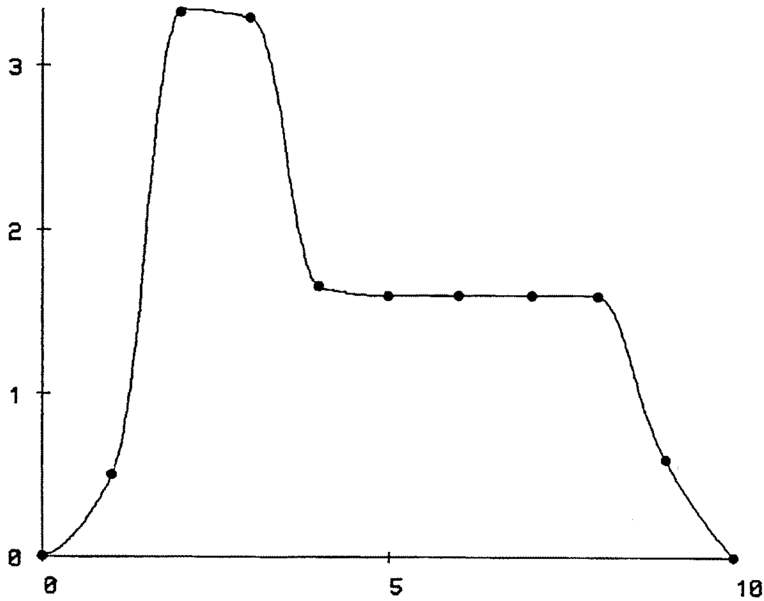

FIGURE 1. Schumaker's quadratic spline interpolating Pruess' data. The si's are defined by (7), (8), and (9) with $\xi = \eta = 0.5$.

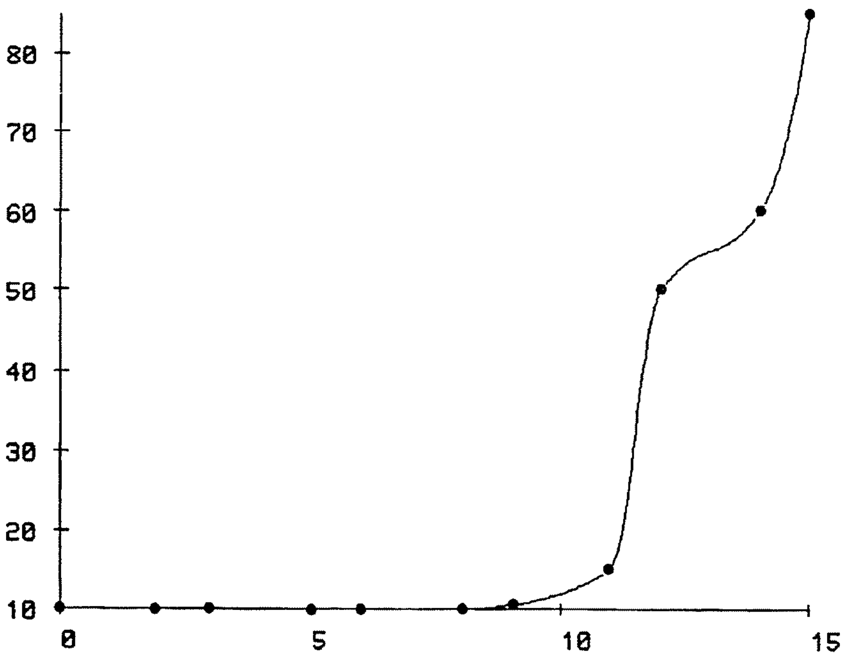

FIGURE 2. Schumaker's quadratic spline interpolating Akima's data. The si's are defined by (7), (8), and (9) with $\xi = \eta = 0.5$.

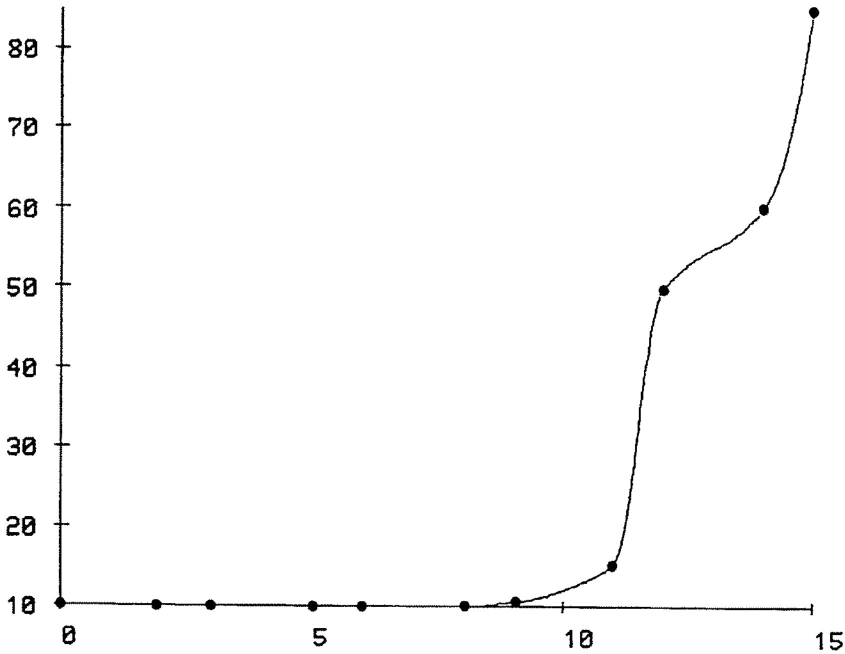

FIGURE 3. Schumaker's quadratic spline interpolating Akima's data. The s_i's are defined by (7), (8), and (9) with $\xi = 0.4$, $\eta = 0.6$.

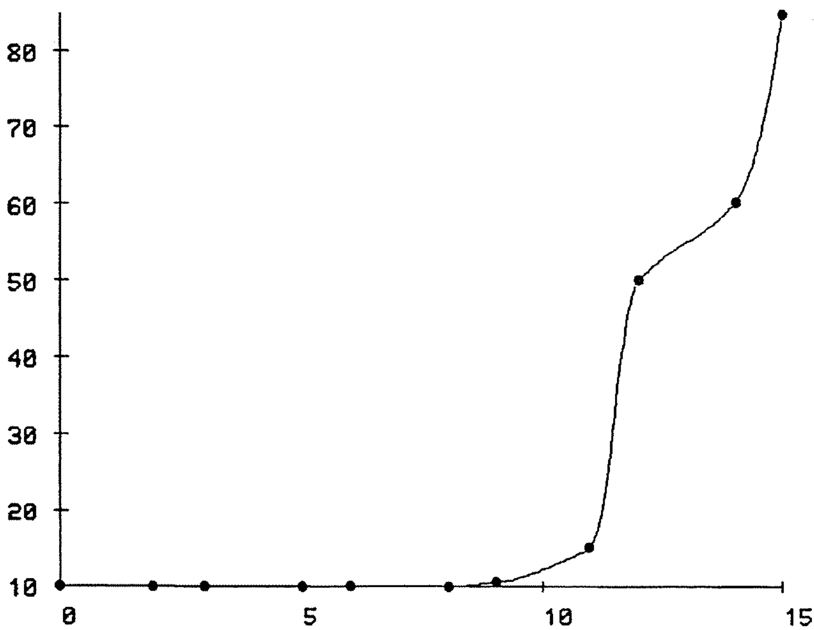

FIGURE 4. Schumaker's quadratic spline interpolating Akima's data. The s_i's are defined by (7), (8), and (9) with $\xi = 0.3$, $\eta = 0.7$.

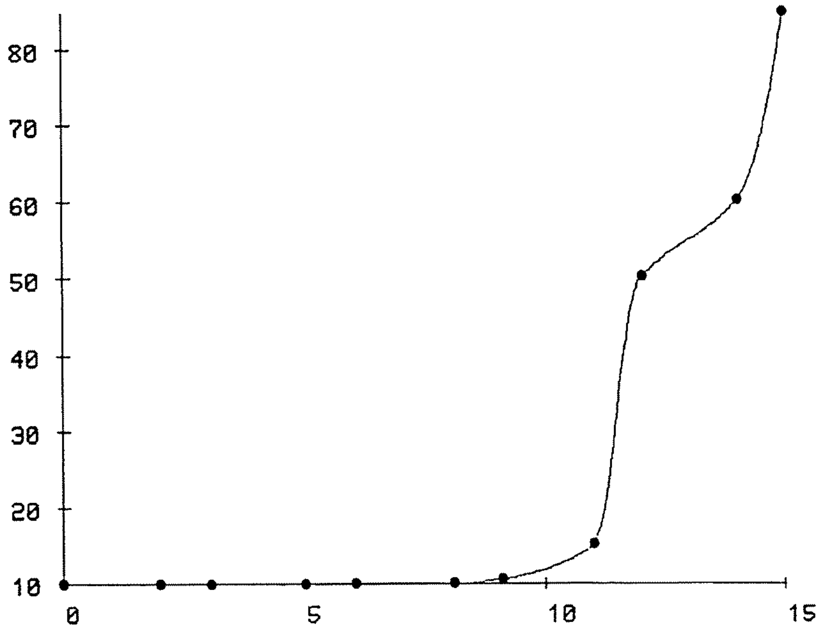

FIGURE 5. Schumaker's quadratic spline interpolating Akima's data. The s_i's are defined by (7), (8), and (9) with $\xi = \eta = 0.5$ except in the interval [x9, x10], where $\xi = 0.3, \eta = 0.7$

Akima's data:

	1	2	3	4	5	6	7	8	9	10	11
x	0	2	3	5	6	8	9	11	12	14	15
y	10	10	10	10	10	10	10.5	15	50	60	85

DISCUSSION

In Schumaker's algorithm, after the derivatives s_i, s_{i+1} have been calculated, a knot in (x_i, x_{i+1}) will be chosen if needed such that the convexity or concavity of the data is conserved. The monotone property of the data may have been violated by this process. When this happens, the values of s_i, s_{i+1} and the location of the knot will have to be adjusted. Our method of defining the s_i's requires no adjustment. They always satisfy condition (6) for monotonicity.

Our method reduces to McAllister and Roulier's method when ξ and η are set to 0.5. This is like a McAllister and Roulier's method with an additional tension parameter that may be built into the method or set by the users to produce the kind of shapes that suit their applications.

Like most methods of this nature, our method is local. Each piece of the quadratic polynomials generated depends on three consecutive data points only. In an interactive computing environment, as soon as three data points have been received by the program implementing this method, pieces of curve can be calculated and displayed on a monitor screen. As a result, the amount of computer memory used is rather small. As was pointed out by Schumaker, even when the curve generation phase is delayed until all the data points are in and all the calculations are done, the basic storage requirement will be the space to store the data, the knots, and the coefficients $\{A_i\}_i$, $\{B_i\}_i$, and $\{C_i\}_i$ of the quadratic polynomials. Since derivatives are matched at the data points, the curve is C^1 for one or more sets of (ξ, η) used in this investigation.

Besides calculating the slopes δ_i's, it takes ten computations (one is a square root) to calculate a Schumaker's s_i. Our method involves equal number of computations but without the square root computation. These attributes combined make our method a good alternative to other shape-preserving quadratic splines.

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